Quaestiones Mathematicae **28**(2005), 425–457.

ORDER CONVERGENCE STRUCTURE ON $C(X)$

Roumen Anguelov

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa.
E-Mail anguelov@scientia.up.ac.za

Jan Harm van der Walt

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa.
E-Mail janharm.vanderwalt@up.ac.za

Abstract. This paper brings together three concepts which have not been related so far, namely, the concept of order convergence, the concept of convergence space and the concept of Hausdorff continuous functions. The order convergence on a poset P, which is generally not a topological convergence, can be studied through the concept of convergence space. Indeed, under certain mild assumptions there exists a convergence structure on P which induces the order convergence. In particular, the result is true for any vector lattice. The primary focus is on the set $C(X)$ of all continuous real functions on a topological space X. The vector lattice $C(X)$ gives a typical example when the order convergence cannot be induced by a topology, thus justifying our interest in the convergence vector structure inducing the order convergence. The completion of the respective convergence vector space is obtained through Hausdorff continuous functions.

Mathematics Subject Classification (2000): 46A19, 46A40, 46E05.

Key words: Order convergence, convergence space, completion.

1. Introduction. The convergence of sequences of functions is one of the fundamental topics associated with functional spaces. Most commonly such convergence is studied within the framework of topology. However, topology does not always give an adequate representation of convergence and there are well known cases where the convergence is not topological, see [5], [8]. The case of order convergence discussed in this paper falls in this category. Hence the more general concepts of convergence structures and convergence spaces are applied.

The recent book [5], which also gives an up to date account on convergence structures, motivates their application through the so called continuous convergence structure on the set $C(X,Y)$ of all continuous functions from the convergence space X to the convergence space Y. Furthermore, it shows convincingly that convergence spaces provide an excellent setting for functional analysis, including the study of duality in vector spaces.

In the present paper we consider convergence spaces with convergence structures defined through partial order, or more precisely, through the so called order convergence. The primary focus is on the set $C(X)$ of all continuous real valued functions defined on a topological space X, where the order convergence is given through the natural partial order which is induced in a point-wise way by the total order on $\mathbb{R}$. The fact that, in general, the order convergence on a poset is not topological is well known, [12], [13]. We will show through examples that this is indeed the case with the order convergence on $C(X)$, see Section 4. We will also show that there exists a convergence structure in the sense of [5] on $C(X)$ which induces sequential convergence identical with the order convergence.

There is a number of natural topologies associated with the set $C(X)$. These include the topology of point-wise convergence (point-open topology), the compact-open topology, the topology of uniform convergence, to mention a few, [11]. For understanding of the place of the order convergence structure introduced in this paper with regard to these well known topologies we should note that the convergence in each one of these topologies is at least as strong as the point-wise convergence. Furthermore, the convergence in the continuous convergence structure discussed in [5] is also stronger than the point-wise convergence. In fact, when X is locally compact the continuous convergence structure on $C(X)$ coincides with the compact-open topology, see [5] [Corollary 1.5.17]. On the other hand we shall see in the sequel that although the order convergence retains some essential features of the uniform convergence, it is nevertheless not point-wise. In this respect, it is appropriate to mention here that in functional analysis we often encounter sequences where the convergence is not point-wise.

The order convergence on a poset is also studied through the concept of the so called order topology which is the finest topology preserving the order convergence, see [6], [12]. In this regard we should mention here the fundamental difference between the order topology and the convergence structure constructed in this paper. Namely, the set of convergent sequences in the order topology is generally larger than the set of order convergent sequences while we, on the other hand, define a convergence structure (a pseudo topology) which gives exactly the same set of convergent sequences as the order convergence.

The paper is organized as follows. Section 2 gives a necessary background on the theory of convergence space. Section 3 deals with the sequential convergence structure induced by the order convergence on a poset, a lattice and a vector lattice. It is shown that, under certain conditions, the order convergence on a lattice P can be induced by a convergence structure on P. This result is interesting particularly in view of the fact that, in general, the sequential order convergence structure on a poset P cannot be induced by a topology. The order convergence on the vector lattice $C(X)$ is discussed in Section 4 where $C(X)$ is proved to be a convergence vector space with convergent sequences exactly as given by the order convergence. However, this convergence vector space is not complete. The completion is obtained through the Hausdorff continuous functions on X. Section 5 is an introduction to the set of Hausdorff continuous functions while the main result concerning the completion of $C(X)$ is discussed in Section 6. Some final remarks are given in the Conclusion. In order to avoid frequent interruptions of the exposition by technical results and lengthy proofs, some technical lemmas with respective proofs as well as the proof of Theorem 26 are given in the Appendix.

2. Convergence structure and convergence space. Since the concepts of convergence structure and convergence space are fundamental for the exposition we will give in this section the basic definitions and some related theorems. For further details as well as the proofs of all statements in this section the reader is referred to [5].

Let $\mathcal{K}$ be a given set on which we shall define a convergence structure. For convenience we recall a few basic concepts related to filters. A filter $\mathcal{A}$ on the set $\mathcal{K}$ is a nonempty collection of subsets of $\mathcal{K}$ which does not contain the empty set and is closed under finite intersections and supersets. A subset $\mathcal{D}$ of a filter $\mathcal{A}$ is called a (filter) basis of $\mathcal{A}$ and $\mathcal{A}$ the filter generated by $\mathcal{D}$ if each set in $\mathcal{A}$ contains a set in $\mathcal{D}$. If $\mathcal{A}$ and $\mathcal{B}$ are filters on $\mathcal{K}$ then $\mathcal{A}$ is called finer than $\mathcal{B}$ and $\mathcal{B}$ coarser than $\mathcal{A}$ if $\mathcal{B} \subseteq \mathcal{A}$ holds.

DEFINITION 1. A mapping λ from the set $\mathcal{K}$ into the power set of the set of filters on $\mathcal{K}$ is called a convergence structure on $\mathcal{K}$ and $(\mathcal{K}, \lambda)$ a convergence space if the following hold for all $f \in \mathcal{K}$:

(i) The filter generated by $\{\{f\}\}$ belongs to $\lambda(f)$.

(ii) For all filters $\mathcal{A}, \mathcal{B} \in \lambda(f)$ the intersection $\mathcal{A} \bigcap \mathcal{B}$ belongs to $\lambda(f)$.

(iii) If $\mathcal{A} \in \lambda(f)$, then $\mathcal{B} \in \lambda(f)$ for all filters $\mathcal{B}$ which are finer than $\mathcal{A}$.

If $\mathcal{A} \in \lambda(f)$ we also say that the filter $\mathcal{A}$ converges to f and write $\mathcal{A} \to f$.

Every topological space generates naturally a convergence space where a filter converges to f whenever it is finer than the neighborhood filter of f. A convergence space is typically not a topological space but a more general structure. This usually happens because given a convergence space the set of filters convergent in it need not coincide with the set of filters convergent in any topology. However, most of the basic topological concepts can be extended to convergence spaces. Below we define continuity and some related concepts.

DEFINITION 2. Let $\mathcal{K}$ and $\mathcal{L}$ be convergence spaces and let $\varphi : \mathcal{K} \to \mathcal{L}$ be a given mapping. For any filter $\mathcal{A}$ on $\mathcal{K}$ the filter generated by the basis $\{\varphi(A) : A \in \mathcal{A}\}$ is called the image filter of $\mathcal{A}$ under φ and is denoted by $\varphi(\mathcal{A})$.

DEFINITION 3. Let $\mathcal{K}$ and $\mathcal{L}$ be convergence spaces. A mapping $\varphi : \mathcal{K} \to \mathcal{L}$ is called continuous at $f \in \mathcal{K}$ if for every filter $\mathcal{A}$ on $\mathcal{K}$ we have $\varphi(\mathcal{A}) \to \varphi(f)$ in $\mathcal{L}$ whenever $\mathcal{A} \to f$ in $\mathcal{K}$. The mapping φ is called continuous if it is continuous at every $f \in \mathcal{K}$, a homeomorphism if it is bijective and both φ and φ^{-1} are continuous and, respectively, an embedding if φ is a homeomorphism onto its codomain.

A convergence structure λ on $\mathcal{K}$ induces in a natural way a sequential convergence on $\mathcal{K}$ as follows. Let $\xi = (\xi_i)_{i \in \mathbb{N}}$ be a sequence on $\mathcal{K}$. The filter generated by the collection of sets $\{\{\xi_i : i \geq n\} : n \in \mathbb{N}\}$ is called the Frechet filter of ξ and is denoted by $\langle \xi \rangle$.

DEFINITION 4. A sequence ξ on a convergence space $(\mathcal{K}, \lambda)$ is said to converge to $f \in \mathcal{K}$ if $\langle \xi \rangle \in \lambda(f)$.

The sequential convergence defined in this way introduces on $\mathcal{K}$ a sequential convergence structure in terms of the following definition.

Definition 5. A mapping σ from the set $\mathcal{K}$ into the power set of the set of all sequences on $\mathcal{K}$ is called a sequential convergence structure and, correspondingly, $(\mathcal{K}, \sigma)$ a sequential convergence space if the following hold for any $f \in \mathcal{K}$:

(i) The constant sequence with the value f belongs to $\sigma(f)$.

(ii) If a sequence belongs to $\sigma(f)$, so does any subsequence of it.

Following the commonly used terminology we will say that the sequence ξ converges to f and also write $\xi \to f$ whenever $\xi \in \sigma(f)$.

The sequential convergence structure σ defined on $\mathcal{K}$ through Definition 4 is called the induced sequential convergence structure.

Definition 6. Let $\mathcal{K}$ and $\mathcal{L}$ be sequential convergence spaces. A mapping $\varphi : \mathcal{K} \to \mathcal{L}$ is called sequentially continuous at $f \in \mathcal{K}$ if for every sequence $\langle \xi \rangle$ on $\mathcal{K}$ we have $\varphi(\xi) \to \varphi(f)$ in $\mathcal{L}$ whenever $\xi \to f$ in $\mathcal{K}$. The mapping φ is called sequentially continuous if it is sequentially continuous at every $f \in \mathcal{K}$.

While every convergence structure on $\mathcal{K}$ induces a sequential convergence structure on $\mathcal{K}$, the converse is not always true, that is, if $(\mathcal{K}, \sigma)$ is a sequential convergence space, then the existence of convergence structure on $\mathcal{K}$ which induces σ cannot be guaranteed. A necessary and sufficient condition is stated in the following theorem.

Theorem 7. *Let $(\mathcal{K}, \sigma)$ be a sequential convergence space. Then there exists a convergence structure λ on $\mathcal{K}$ inducing σ if and only if for any two sequences ξ and η on $\mathcal{K}$ and $f \in \mathcal{K}$ the following hold:*

(i) *If $\xi \to f$ and $\langle \eta \rangle = \langle \xi \rangle$ then $\eta \to f$.* (1)

(ii) *If $\xi \to f$ and $\eta \to f$ then $\xi \diamond \eta \to f$, where $\xi \diamond \eta$ denotes the trivial mixture of ξ and η, that is, $(\xi \diamond \eta)_{2n-1} = \xi_n$ and $(\xi \diamond \eta)_{2n} = \eta_n$ for all $n \in \mathbb{N}$.* (2)

Following the terminology adopted in [5] a sequential convergence space satisfying the conditions (1) and (2) is called an FS-space. The following concepts will also be used in the sequel.

Definition 8. A convergence space $(\mathcal{K}, \lambda)$ is called first countable if, for each filter converging to an element f, there exists a coarser filter with a countable basis which still converges to f.

Definition 9. A convergence space $(\mathcal{K}, \lambda)$ is called sequentially determined if the following hold:

(i) $(\mathcal{K}, \lambda)$ is first countable.

(ii Given $f \in \mathcal{K}$ a filter $\mathcal{A}$ on $\mathcal{K}$ with a countable basis converges to f whenever each sequence ξ which is finer than $\mathcal{A}$, that is, $\mathcal{A} \subseteq \langle \xi \rangle$, converges to f.

The convergence structure inducing the sequential convergence structure σ in an FS-space $(\mathcal{K}, \sigma)$ is generally not unique. However, there is a unique sequentially determined convergence structure inducing σ and the following theorem gives a general way of constructing it.

THEOREM 10. *Given an FS-space $(\mathcal{K}, \sigma)$ a convergence structure $\gamma(\sigma)$ which induces σ can be defined on $\mathcal{K}$ as follows:*

> *A filter $\mathcal{A}$ converges to f or $\mathcal{A} \in \gamma(\sigma)(f)$ if and only if there is a coarser filter $\mathcal{B}$ with a countable basis with the property that $\xi \to f$ for all sequences ξ which are finer than $\mathcal{B}$.*

Furthermore, $\gamma(\sigma)$ is the unique sequentially determined convergence structure on $\mathcal{K}$ which induces the sequential convergence structure σ.

The concept of convergence vector space given below combines the concepts of convergence space and vector space in a similar way as the concept of topological vector space brings together the concepts of topological space and vector space.

DEFINITION 11. A convergence structure λ on a real vector space $\mathcal{K}$ is called a vector space convergence structure and $(\mathcal{K}, \lambda)$ convergence vector space if addition and scalar multiplication are continuous.

In the above definition the addition is a mapping defined on the Tychonoff product space $\mathcal{K} \times \mathcal{K}$ and the scalar multiplication is a mapping defined on the Tychonoff product space $\mathbb{R} \times \mathcal{K}$. Let us recall the definition of the Tychonoff product of convergence spaces. Given a family of convergence spaces $(X_i)_{i \in \mathcal{I}}$ the Tychonoff product convergence structure on $\prod_{i \in \mathcal{I}} X_i$ is defined through the projection mappings $p_i : \prod_{j \in \mathcal{I}} X_j \to X_i$, $i \in \mathcal{I}$, as follows:

$$\mathcal{A} \to f \text{ in } \prod_{i \in \mathcal{I}} X_i \iff p_i(\mathcal{A}) \to p_i(f) \text{ in } X_i,\ i \in \mathcal{I}. \tag{3}$$

The vector space convergence structure on $\mathcal{K}$ introduces in a natural way a uniform convergence structure - a generalization of the concept of uniformity associated with the uniform spaces. Cauchy filter and Cauchy sequence as well as the related concept of completeness are defined on a convergence vector space as follows.

DEFINITION 12. A filter $\mathcal{A}$ on a convergence vector space $\mathcal{K}$ is called a Cauchy filter if the filter $\mathcal{A} - \mathcal{A}$ converges to zero (the additive neutral element). A sequence ξ on $\mathcal{K}$ is called a Cauchy sequence if $\langle \xi \rangle$ is a Cauchy filter, that is, $\langle \xi \rangle - \langle \xi \rangle$ converges to zero (the additive neutral element).

DEFINITION 13. A convergence vector space $\mathcal{K}$ is called complete if every Cauchy filter converges and it is called sequentially complete if every Cauchy sequence converges.

The completion of a convergence vector space is in general a complicated issue. Part of the problem is in the following. For a convergence vector space $\mathcal{K}$ one can construct a completion $\tilde{\mathcal{K}}$ of the uniform convergence structure very much in the same way as in the case of uniform topological space and this completion is unique. The algebraic operations are extended on $\tilde{\mathcal{K}}$ in a natural way so that $\tilde{\mathcal{K}}$ becomes a convergence vector space. However, the subspace convergence structure induced from $\tilde{\mathcal{K}}$ to $\mathcal{K}$ may differ from the original convergence structure on $\mathcal{K}$. It was pointed out in [5] [Chapter 2.3] that in general there are three different completion theories, namely, for uniform convergence spaces, for convergence groups and for convergence vector spaces. The completion of convergence vector space is particularly discussed in [7].

3. Order convergence and the corresponding convergence structure on a poset and a vector lattice. Let P be a poset with a partial order $\leq$.

DEFINITION 14. A sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$ on P is said to order converge to $f \in P$ if there exist on P an increasing sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and a decreasing sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ such that

$$\alpha_n \leq \xi_n \leq \beta_n,\ n \in \mathbb{N},$$

and

$$f = \sup_{n\in\mathbb{N}} \alpha_n = \inf_{n\in\mathbb{N}} \beta_n.$$

It is easy to see that the order convergence on a poset P satisfies the conditions stated in Definition 5. Hence it introduces on the poset a sequential convergence structure which we denote by σ_o, that is, for any sequence ξ on P and $f \in P$

$$\xi \in \sigma_o(f) \Longleftrightarrow \xi \text{ order converges to } f\ . \tag{4}$$

It is well known that, in general, the order convergence cannot be derived from a topology. More precisely, if P is a topological space the class of convergent sequences with respect to the topology defines a sequential convergence structure, see Definition 5, which satisfies two additional properties, namely,

Urysohn property: A sequence converges to f whenever every subsequence has a subsequence which converges to f. (5)

Diagonal property: Let for every $n \in \mathbb{N}$ the sequence $(p_{nm})_{m\in\mathbb{N}}$ converge to ξ_n. If the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to f then there exists a mapping $k : \mathbb{N} \to \mathbb{N}$ such that the sequence $(p_{n\,k(n)})_{n\in\mathbb{N}}$ converges to f. (6)

In general, the sequential order convergence structure σ_o does not have these two properties. In the next section we will give an example showing that the sequential order convergence structure σ_o on $C(X)$ does not satisfy the Urysohn property. An example that σ_o also violates the diagonal property on $C(X)$ is given in [15]. Hence the sequential convergence structure σ_o on P cannot, in general, be induced by a topology. Then it is an interesting question if σ_o can be induced by a convergence structure on P. The following theorem shows that at least one of the conditions in Theorem 7 is satisfied under very general assumptions for P.

THEOREM 15. *Let the poset P be a lattice. The sequential convergence space (P, σ_o) satisfies property (1), that is, for any two sequences ξ and η on P, if $\xi \to f$ and $\langle \eta \rangle = \langle \xi \rangle$ then $\eta \to f$.*

Proof. Let ξ and η be sequences on P where $\xi \to f$ and $\langle \xi \rangle = \langle \eta \rangle$. Since the sequence ξ order converges to f there exist sequences $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}}$ with the properties specified in Definition 14. First let us note that for every $n \in \mathbb{N}$ the elements α_n and β_n are respectively lower and upper bounds of the set $\{\xi_i : i \geq n\}$. Indeed, we have

$$\alpha_n \leq \alpha_i \leq \xi_i \leq \beta_i \leq \beta_n, \ i \geq n.$$

Furthermore, since $\{\xi_i : i \geq n\} \in \langle \xi \rangle = \langle \eta \rangle$ there exists $k_n \in \mathbb{N}$ such that

$$\{\eta_j : j \geq k_n\} \subseteq \{\xi_i : i \geq n\}$$

Then α_n and β_n are respectively lower and upper bounds of the set $\{\eta_j : j \geq k_n\}$. Hence we can construct inductively an increasing sequence of naturals $k_1, k_2, k_3, ...$ such that

$$\alpha_n \leq \eta_j \leq \beta_n, j \geq k_n, \ n \in \mathbb{N}. \tag{7}$$

Now we can define two new sequences $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ as follows:

$$\begin{aligned} a_j &= \inf\{\eta_1, ..., \eta_{k_1 -1}, \alpha_1\}, \ j = 1, 2, ..., k_1 - 1 \\ a_j &= \alpha_n, \ j = k_n, k_n + 1, ..., k_{n+1} - 1, \ n = 1, 2, ... \end{aligned}$$

$$\begin{aligned} b_j &= \sup\{\eta_1, ..., \eta_{k_1 -1}, \beta_1\}, \ j = 1, 2, ..., k_1 - 1 \\ b_j &= \beta_n, \ j = k_n, k_n + 1 ..., k_{n+1} - 1, \ n = 1, 2, ... \end{aligned}$$

Clearly, $(a_j)_{j\in\mathbb{N}}$ is an increasing sequence while $(b_j)_{j\in\mathbb{N}}$ is a decreasing sequence. From the inequality (7) it follows that

$$a_j \leq \eta_j \leq b_j, \ j \in \mathbb{N}.$$

We also have

$$\begin{aligned} \sup_{j\in\mathbb{N}} a_j &= \sup_{n\in\mathbb{N}} \alpha_n = f \\ \inf_{j\in\mathbb{N}} b_j &= \inf_{n\in\mathbb{N}} \beta_n = f. \end{aligned}$$

Hence it follows from Definition 14 that the sequence η order converges to f. $\square$

The sequential convergence space (P,σ_o) is an FS-space only if condition (2) is also satisfied. The characterization of the posets on which the order convergence structure σ_o satisfies condition (2) is an open problem. If this condition is satisfied then according to Theorem 7 there exists on P a convergence structure inducing σ_o. In such a case we will define explicitly a first countable convergence structure on P which induces σ_o. As usual for arbitrary $f,g\in P$, $f\le g$, by $[f,g]$ we denote the interval with end points f and g, that is,

$$[f,g]=\{\phi\in P: f\le\phi\le g\}.$$

Let λ_o be a mapping from P into the power set of the filters on P defined through

$$\mathcal{A}\in\lambda_o(f)\iff\left\{\begin{array}{l}\text{there exists a coarser filter } \mathcal{B} \text{ generated by a}\\ \text{basis of the form } \{[\alpha_n,\beta_n]:n\in\mathbb{N}\} \text{ where}\\ \bullet\ (\alpha_n)_{n\in\mathbb{N}} \text{ is an increasing sequence on } P\\ \bullet\ (\beta_n)_{n\in\mathbb{N}} \text{ is a decreasing sequence on } P\\ \bullet\ f=\sup\limits_{n\in\mathbb{N}}\alpha_n=\inf\limits_{n\in\mathbb{N}}\beta_n\end{array}\right. \qquad (8)$$

THEOREM 16. *Let the poset P be a lattice and let (P,σ_o) be an FS-space. Consider on P the mapping λ_o given in (8). Then*

(i) λ_o is a convergence structure;

(ii) if a filter $\mathcal{A}\in\lambda_o(f)$ has a countable basis $\{A_1,A_2,\ldots\}$ where A_1 is (order) bounded and $A_1\supseteq A_2\supseteq\ \ldots$, then there exist an increasing sequence $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ and a decreasing sequence $\beta=(\beta_n)_{n\in\mathbb{N}}$ such that

$$\alpha_n\le\phi\le\beta_n,\ \phi\in A_n,\ n\in\mathbb{N}$$
$$f=\sup_{n\in\mathbb{N}}\alpha_n=\inf_{n\in\mathbb{N}}\beta_n;$$

(iii) λ_o induces on P the sequential convergence structure σ_o.

Proof. (i) Conditions (i) and (iii) of Definition 1 are obvious. We will prove (ii). Let $\mathcal{A}^{(1)},\mathcal{A}^{(2)}\in\lambda_o(f)$. Then there exist sequences $\alpha^{(1)}=(\alpha_n^{(1)})_{n\in\mathbb{N}}$, $\beta^{(1)}=(\beta_n^{(1)})_{n\in\mathbb{N}}$ and $\alpha^{(2)}=(\alpha_n^{(2)})_{n\in\mathbb{N}}$, $\beta^{(2)}=(\beta_n^{(2)})_{n\in\mathbb{N}}$ which can be associated respectively with the filters $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ in terms of the definition of λ_o, see (8). Denote $\alpha_n=\inf\{\alpha_n^{(1)},\alpha_n^{(2)}\}$, $\beta_n=\sup\{\beta_n^{(1)},\beta_n^{(2)}\}$, $n\in\mathbb{N}$. It is easy to see that the filter $\mathcal{B}$ generated by the base $\{[\alpha_n,\beta_n]:n\in\mathbb{N}\}$ is coarser than the filter $\mathcal{A}=\mathcal{A}^{(1)}\cap\mathcal{A}^{(2)}$. Furthermore, since both $(\alpha^{(1)})_{n\in\mathbb{N}}$ and $(\alpha^{(2)})_{n\in\mathbb{N}}$ are increasing the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is also increasing. Similarly, the sequence $(\beta_n)_{n\in\mathbb{N}}$ is decreasing. Hence it remains to show that $f=\sup\limits_{n\in\mathbb{N}}\alpha_n=\inf\limits_{n\in\mathbb{N}}\beta_n$. We will prove the first equality since the second one is proved in a similar way. It follows directly from Definition 14 that the sequences $\alpha^{(1)}$ and $\alpha^{(2)}$ order converge to f. Due to the assumption that (P,σ_o) is an FS-space the trivial mixture $\alpha^{(1)}\diamond\alpha^{(2)}$ also order converges to f. Then there exist an increasing sequence $\tilde{\alpha}=(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ and a decreasing sequence $\tilde{\beta}=(\tilde{\beta}_n)_{n\in\mathbb{N}}$ such that

$$\tilde{\alpha}_{2n-1}\le\alpha_n^{(1)}\le\tilde{\beta}_{2n-1}\ ,\quad \tilde{\alpha}_{2n}\le\alpha_n^{(2)}\le\tilde{\beta}_{2n}\ ,\quad n\in\mathbb{N}, \tag{9}$$

$$f=\sup_{n\in\mathbb{N}}\tilde{\alpha}_n=\inf_{n\in\mathbb{N}}\tilde{\beta}_n. \tag{10}$$

Then we have

$$\alpha_n=\inf\{\alpha_n^{(1)},\alpha_n^{(2)}\}\ge\tilde{\alpha}_{2n-1}$$
$$\alpha_n=\inf\{\alpha_n^{(1)},\alpha_n^{(2)}\}\le\tilde{\beta}_{2n}.$$

Using (10) and the monotonicity of the involved sequences the above inequalities imply $f=\sup_{n\in\mathbb{N}}\alpha_n$, which completes the proof of part (i) of the theorem.

(ii) Let $\mathcal{A}$ be a filter as specified in the theorem. Then there exists a coarser filter $\mathcal{B}$ generated by a basis of the form $\{[\tilde{\alpha}_n,\tilde{\beta}_n]:n\in\mathbb{N}\}$, where the sequences $\tilde{\alpha}=(\tilde{\alpha}_n)_{n\in\mathbb{N}}$, $\tilde{\beta}=(\tilde{\beta}_n)_{n\in\mathbb{N}}$ satisfy the conditions given in the definition of λ_o, see (8). Since $\mathcal{B}$ is coarser than $\mathcal{A}$, for every $n\in\mathbb{N}$ there exists k_n such that $A_{k_n}\subseteq[\tilde{\alpha}_n,\tilde{\beta}_n]$. Let a and b be respectively lower and upper bounds of the set A_1. Then the required sequences α and β can be constructed as stated below

$$\begin{aligned}\alpha_j &= \inf\{a,\tilde{\alpha}_1\},\ j=1,...,k_1-1,\\ \alpha_j &= \tilde{\alpha}_n,\ j=k_n,k_n+1,...,k_{n+1}-1,\ n=1,2,...\end{aligned}$$

$$\begin{aligned}\beta_j &= \sup\{b,\tilde{\beta}_1\},\ j=1,...,k_1-1,\\ \beta_j &= \tilde{\beta}_n,\ j=k_n,k_n+1,...,k_{n+1}-1,\ n=1,2,...\end{aligned}$$

(iii) We need to proof that a sequence ξ on P order converges to $f\in P$ if and only if its Frechet filter $\langle\xi\rangle$ converges to f in the convergence structure λ_o. The implication in one direction, namely that the order convergence of a sequence implies that its Frechet filter converges in λ_o follows directly from the way λ_o is defined, see (8). We will prove the converse implication. Assume that for a given sequence ξ its Frechet filter $\langle\xi\rangle$ converges to f in λ_o. Then we can associated with $\langle\xi\rangle$ a coarser filter $\mathcal{B}$ and respective sequences $\tilde{\alpha}=(\tilde{\alpha}_n)_{n\in\mathbb{N}}$, $\tilde{\beta}=(\tilde{\beta}_n)_{n\in\mathbb{N}}$ satisfying the conditions given in (8). The Frechet filter $\langle\xi\rangle$ is generated by the basis $\{\{\xi_m:m\ge n\}:n\in\mathbb{N}\}$. Using that $\mathcal{B}$ is coarser than $\langle\xi\rangle$ there exists $k\in\mathbb{N}$ such that $\{\xi_m:m\ge k\}\subseteq[\tilde{\alpha}_1,\tilde{\beta}_1]$. Since the sequence ξ has only finite number of elements outside the interval $[\tilde{\alpha}_1,\tilde{\beta}_1]$ it is bounded. Now we can apply the statement proved in (ii) to the filter $\langle\xi\rangle$. There exist an increasing sequence $\alpha=(\alpha_n)_{n\in\mathbb{N}}$ and a decreasing sequence $\beta=(\beta_n)_{n\in\mathbb{N}}$ such that

$$\alpha_n\le\xi_m\le\beta_n,\ m\ge n,\ n\in\mathbb{N}$$
$$f=\sup_{n\in\mathbb{N}}\alpha_n=\inf_{n\in\mathbb{N}}\beta_n.$$

This implies that ξ order converges to f. □

In the sequel we will refer to λ_o as the order convergence structure on the poset P and to the space (P, λ_o) as order convergence space. From the definition of λ_o it follows immediately that it is first countable. However the characterization of the posets for which it is also sequentially determined is an open problem. We should note that the sequentially determined convergence structure $\gamma(\sigma_o)$ given through Theorem 10 is the coarsest (maximal) convergence structure on P inducing σ_o and we have

$$\lambda_o(f) \subseteq \gamma(\sigma_o)(f), \ f \in P.$$

It can be shown that λ_o is the finest (minimal) convergence structure which induces σ_o and satisfies the "squeeze" property, namely, if the sequences ξ and η with $\xi \leq \eta$ both converge to f then so does the filter generated by the basis $\{\{p \in P : \xi_n \leq p \leq \eta_n\} : n \in \mathbb{N}\}$.

As mentioned already the sequential convergence structure σ_o does not satisfy the diagonal property, see (3). Lemma 36 which is given in the Appendix states a result for monotone sequences which is similar to the diagonal property. This result is useful in situations requiring the diagonal property and will indeed be applied in the sequel, namely in Theorem 27.

The next theorem deals with the special case when the poset P is a vector lattice or Riesz space.

THEOREM 17. *Let P be a vector lattice. Then (P, σ_o) is an FS-space. Furthermore, if P is an Archimedean vector lattice then the convergence structure λ_o given in (8) is a vector space convergence structure and (P, λ_o) is a convergence vector space.*

Proof. We need to show that σ_o satisfies conditions (1) and (2). Condition (1) follows from Theorem 15 since P is a lattice. We will prove condition (2). Let the sequences ξ and η both order converge to f and let the sequences $(\alpha_n^{(1)})_{n\in\mathbb{N}}$, $(\beta_n^{(1)})_{n\in\mathbb{N}}$ and $(\alpha_n^{(2)})_{n\in\mathbb{N}}$, $(\beta_n^{(2)})_{n\in\mathbb{N}}$ be the sequences associated respectively with ξ and η in terms of Definition 14. Consider the sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ defined by

$$\alpha_{2k-1} = \alpha_{2k} = \inf\{\alpha_k^{(1)}, \alpha_k^{(2)}\}, \ k \in \mathbb{N},$$
$$\beta_{2k-1} = \beta_{2k} = \sup\{\beta_k^{(1)}, \beta_k^{(2)}\}, \ k \in \mathbb{N}.$$

Then the trivial mixture $\xi \diamond \eta$ consisting of alternating elements of ξ and η satisfies

$$\alpha_n \leq (\xi \diamond \eta)_n \leq \beta_n, \ n \in \mathbb{N}.$$

It was shown in [9] [Theorem 15.3] that if P is a vector lattice then for sequences $(\alpha_n^{(1)})_{n\in\mathbb{N}}$, $(\alpha_n^{(2)})_{n\in\mathbb{N}}$ as given here we have that $(\inf\{\alpha_k^{(1)}, \alpha_k^{(2)}\})_{k\in\mathbb{N}}$ is an increasing sequence and

$$\sup_{k\in\mathbb{N}}(\inf\{\alpha_k^{(1)}, \alpha_k^{(2)}\}) = \inf\{\sup_{k\in\mathbb{N}} \alpha_k^{(1)}, \sup_{k\in\mathbb{N}} \alpha_k^{(2)}\} = f.$$

Hence the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is increasing and $f = \sup_{n\in\mathbb{N}} \alpha_n$. In a similar way, the sequence $(\beta_n)_{n\in\mathbb{N}}$ is decreasing and $f = \inf_{n\in\mathbb{N}} \beta_n$. Then it follows from Definition 14 that the trivial mixture of the sequences ξ and η converges to f. Hence condition (2) is satisfied and (P, σ_o) is an FS-space.

Let now P be an Archimedean vector lattice. It is well know that in this case the vector space operations are sequentially continuous with respect to the order convergence, see [16] [Theorem 10.2(iii)] and [9] [Chapter 2.16]. Sequential continuity, in general, does not imply continuity. Moreover, the convergence space (P, λ_o) is first countable but whether or not it is sequentially determined is an open problem. Hence theorems like [5] [Theorem 1.6.14] are not applicable. That is why here we will give a direct proof of the continuity of the vector space operations on P.

For convenience denote by $d : P \times P \to P$ the addition mapping, that is, $d(a_1, a_2) = a_1 + a_2$, $a_1, a_2 \in P$. Let $p_1, p_2 : P \times P \to P$ be the projection mappings on $P \times P$ about the first coordinate and about the second coordinate, respectively. Assume that $\mathcal{A}$ is a filter on $P \times P$ which converges to $(a_1, a_2) \in P \times P$ in the product convergence structure on $P \times P$, see (3). This means that $p_i(\mathcal{A}) \in \lambda_o(a_i)$, $i = 1, 2$. From the definition of λ_o, see (8), it follows that there exists filters $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\mathcal{B}_i \subseteq p_i(\mathcal{A})$, $i = 1, 2$, which are respectively generated by bases of the form

$$\{[\alpha_n^{(i)}, \beta_n^{(i)}] : n \in \mathbb{N}\},\ \ i = 1, 2,$$

where the sequences $(\alpha_n^{(i)})_{n\in\mathbb{N}}$, $i = 1, 2$, are increasing, the sequences $(\beta_n^{(i)})_{n\in\mathbb{N}}$, $i = 1, 2$, are decreasing and

$$a_i = \sup_{n\in\mathbb{N}} \alpha_n^{(i)} = \inf_{n\in\mathbb{N}} \beta_n^{(i)}\ ,\quad i = 1, 2.$$

It is easy to see that the filter $\mathcal{D}$ on $P \times P$ generated by a basis of the form

$$\{[\alpha_n^{(1)}, \beta_n^{(1)}] \times [\alpha_n^{(2)}, \beta_n^{(2)}] : n \in \mathbb{N}\},$$

is coarser than $\mathcal{A}$. Furthermore, since $p_i(\mathcal{D}) = \mathcal{B}_i$, $i = 1, 2$, the filter $\mathcal{D}$ converges to (a_1, a_2). The image filter $d(\mathcal{D})$ is generated by the basis

$$\{[\alpha_n^{(1)} + \alpha_n^{(2)}, \beta_n^{(1)} + \beta_n^{(2)}] : n \in \mathbb{N}\}.$$

The sequential continuity of the operation addition implies that the increasing sequence $(\alpha_n^{(1)} + \alpha_n^{(2)})_{n\in\mathbb{N}}$ order converges to $a_1 + a_2$. Hence

$$a_1 + a_2 = \sup_{n\in\mathbb{N}} (\alpha_n^{(1)} + \alpha_n^{(2)}).$$

In a similar way for the decreasing sequence $(\beta_n^{(1)} + \beta_n^{(2)})_{n\in\mathbb{N}}$ we have

$$a_1 + a_2 = \inf_{n\in\mathbb{N}} (\beta_n^{(1)} + \beta_n^{(2)}).$$

Therefore, the filter $d(\mathcal{D})$ converges to $a_1 + a_2$ with respect to the convergence structure λ_o on P, see (8). Hence the filter $d(A)$, being finer than $d(\mathcal{D})$, also converges to $a_1 + a_2$. Thus the operation addition is continuous.

The continuity of the scalar multiplication is proved by using similar arguments. □

The next theorem gives a useful characterization of the Cauchy sequences on the convergence vector space (P, λ_o).

THEOREM 18. *Let P be a given Archimedean vector lattice such that (P, λ_o) is a convergence vector space and let ξ be a sequence on P. Then ξ is a Cauchy sequence if and only if there exist a decreasing sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ such that*

$$\xi_m - \xi_k \le \beta_n,\ m, k \ge n,\ n \in \mathbb{N}, \tag{11}$$
$$\inf_{n\in\mathbb{N}} \beta_n = \mathcal{O},$$

where $\mathcal{O}$ is the additive neutral element of the vector space P.

Proof. According to Definition 12, the sequence ξ is Cauchy if and only if the filter $\langle\xi\rangle - \langle\xi\rangle$ converges to $\mathcal{O}$. It is easy to see that the filter $\langle\xi\rangle - \langle\xi\rangle$ is generated by the basis

$$\{\{\xi_m - \xi_k : m, k \ge n\} : n \in \mathbb{N}\}.$$

Let ξ be a Cauchy sequence. Then $\langle\xi\rangle - \langle\xi\rangle \in \lambda_o(\mathcal{O})$ and one can easily obtain from (8) that ξ is bounded. Then the existence of the sequence β follows from Theorem 16(ii). To prove the inverse implication we should note that the inequality (11) is equivalent to

$$-\beta_n \le \xi_m - \xi_k \le \beta_n,\ m, k \ge n,\ n \in \mathbb{N}.$$

Then the fact that $\langle\xi\rangle - \langle\xi\rangle$ converges to $\mathcal{O}$ follows directly from the definition of λ_o, see (8). □

We should note that the characterization of Cauchy sequences in Theorem 18 coincides with the definition of order Cauchy sequence on a vector lattice which is given in [16] [Chapter 4]. Clearly the concept of Cauchy sequence cannot be formulated within the realm of sequences only. The definition in [16] is given without any connection to a particular uniform structure. We believe that it is a an advantage with regard to both clarity and applicability to use a concept of Cauchy sequence given through the uniform convergence structure naturally associated with a vector space convergence structure as is the approach adopted in this paper.

4. Order convergence structure on $C(X)$. Consider the set $C(X)$ of all continuous real functions defined on a given topological space X with a point-wise defined partial order, that is, for $f, g \in C(X)$

$$f \le g \iff f(x) \le g(x),\ x \in X. \tag{12}$$

Then $(C(X), \sigma_o)$, where σ_o is the sequential convergence structure on $C(X)$ given by the order convergence with respect to the partial order (12), is a sequential convergence space.

Let us note that any finite subset of $C(X)$ has both supremum and infimum which are respectively the point-wise supremum and infimum. Thus $C(X)$ is a lattice. However, the existence of supremum and infimum of infinite sets cannot be guaranteed. In particular the supremum and infimum in the Definition 14 might not exist. Furthermore, when the supremum and/or infimum exist they are not necessarily equal to the point-wise supremum and/or infimum of the respective sequences of functions as the later ones might not be continuous functions at all. This is demonstrated on the following example which also shows that the order convergence on $C(X)$ is not point-wise.

EXAMPLE 19. Take $X = \mathbb{R}$ with the usual topology on $\mathbb{R}$ and consider the sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ given by

$$\varphi_n(x) = \begin{cases} 1 - n|x| & \text{if } x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Let z denote the constant zero function, that is, $z(x) = 0$, $x \in \mathbb{R}$. Then z is the largest lower bound of the set $\{\varphi_n : n \in \mathbb{N}\}$ in $C(\mathbb{R})$ with respect to the partial order (12), that is, $z = \inf_{n\in\mathbb{N}} \varphi_n$. Using also that $(\varphi_n)_{n\in\mathbb{N}}$ is a decreasing sequence and taking $\alpha_n = z$ and $\beta_n = \varphi_n$, $n \in \mathbb{N}$, we obtain from Definition 14 that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ order converges to z. Note that z is not a point-wise limit of $(\varphi_n)_{n\in\mathbb{N}}$ and that the point-wise limit is actually not a continuous function.

The above example shows that order convergence does not imply point-wise convergence. The converse is also true, point-wise convergence does not imply in general order convergence. However, under some assumptions for X, e.g. X compact, and for certain classes of sequences, e.g. bounded sequences, point-wise convergence implies order convergence.

We will show next that the order convergence on $C(X)$ is not topological. As mentioned already, the class of the convergent sequences in any topology satisfies the Urysohn property, see (3). The following example shows that the order convergence on $C(X)$ does not have the Urysohn property.

EXAMPLE 20. Consider $C(\mathbb{R})$. Let the rational numbers in the interval $[0, 1]$ be arranged in a sequence $q_1, q_2, \ldots$ and let

$$\psi_n(x) = \varphi_n(x - q_n),\ x \in \mathbb{R},\ n \in \mathbb{N},$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is the sequence given in (13). We will show that every subsequence of the sequence $(\psi_n)_{n\in\mathbb{N}}$ has a further subsequence which converges to the constant zero function denoted by z. Let $(\psi_{n_k})_{k\in\mathbb{N}}$ be any subsequence of $(\psi_n)_{n\in\mathbb{N}}$. Since the interval $[0, 1]$ is a compact subset of $\mathbb{R}$ the sequence $(q_{n_k})_{k\in\mathbb{N}}$ has a convergent subsequence. Let the subsequence $(q_{n_{k_i}})_{i\in\mathbb{N}}$ converge to $q \in [0, 1]$. Denote

$$\varepsilon_i = \max\{|q_{n_{k_j}} - q| : j \geq i\},\ i \in \mathbb{N}.$$

Clearly the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ is decreasing and converges to 0. Consider

$$\beta_i(x)=\begin{cases} 0 & \text{if} \quad x\le q-\varepsilon_i-\frac{1}{i} \\ 1+i(x-q+\varepsilon_i) & \text{if} \quad x\in\left(q-\varepsilon_i-\frac{1}{i},q-\varepsilon_i\right) \\ 1 & \text{if} \quad x\in[q-\varepsilon_i,q+\varepsilon_i] \\ 1-i(x-q-\varepsilon_i) & \text{if} \quad x\in\left(q+\varepsilon_i,q+\varepsilon_i+\frac{1}{i}\right) \\ 0 & \text{if} \quad x\ge q+\varepsilon_i+\frac{1}{i}. \end{cases}$$

The sequence $(\beta_i)_{i\in\mathbb{N}}$ is decreasing and its infimum in $C(\mathbb{R})$ is the constant zero function z. Then the inequalities

$$0\le\psi_{n_{k_i}}(x)\le\beta_i(x),\ x\in\mathbb{R},\ i\in\mathbb{N},$$

imply that $(\psi_{n_{k_i}})_{i\in\mathbb{N}}$ order converges to z. Thus, an arbitrary subsequence of the sequence $(\psi_n)_{n\in\mathbb{N}}$ has a subsequence which converges to z. If the Urysohn property is satisfied the sequence $(\psi_n)_{n\in\mathbb{N}}$ should also converge to z. We will show that this is not true. Let us assume that $(\psi_n)_{n\in\mathbb{N}}$ order converges to z. Then according to Definition 14 there exists a decreasing sequence $(\tilde{\beta}_n)_{n\in\mathbb{N}}$ on $C(\mathbb{R})$ with an infimum equal to z such that $\tilde{\beta}_n(x)\ge\psi_n(x)$, $x\in\mathbb{R}$, $n\in\mathbb{N}$. Let $n\in\mathbb{N}$. For any $j\ge n$ we have

$$\tilde{\beta}_n(q_j)\ge\tilde{\beta}_j(q_j)\ge\psi_j(q_j)=1.$$

Since the set $\{q_j : j\ge n\}$ is dense in the interval $[0,1]$ and the function β_n is continuous on $\mathbb{R}$ the above inequality implies that

$$\tilde{\beta}_n(x)\ge 1,\ x\in[0,1],\ n\in\mathbb{N}.$$

Therefore the infimum of the sequence $(\tilde{\beta}_n)_{n\in\mathbb{N}}$ is not the constant zero function z. Hence the sequence $(\psi_n)_{n\in\mathbb{N}}$ does not converge to z.

As shown by Example 20 it is not possible to have on $C(X)$ a topology which induces the order convergence. However, using the discussion in Section 3 we can show that there exists a convergence structure on $C(X)$ which induces the order convergence and even construct such convergence structure. Indeed, since $C(X)$ is a real vector lattice with the point-wise defined addition and scalar multiplication and order as given in (12), see [16] [Example 4.2(6)], it follows from Theorem 17 that $(C(X),\sigma_o)$ is an FS-space. Hence, there exists a convergence structure on $C(X)$ which induces σ_o. Furthermore, using that $C(X)$ is an Archimedean vector lattice we have that $(C(X),\lambda_o)$ is a convergence vector space, where the convergence structure λ_o is given by (8) and induces σ_o.

It is shown in [5] [Proposition 3.6.5] that for a first countable convergence vector space the completeness and the sequential completeness are equivalent. Since the convergence vector space $(C(X),\lambda_o)$ is first countable it is sufficient to use sequential arguments with regard to its completeness. However we should recall that the Cauchy sequences are defined through λ_o not just σ_o. Consider the following example.

EXAMPLE 21. The sequence $\phi = (\phi_n)_{n\in\mathbb{N}}$ on $C(\mathbb{R})$ is given by

$$\phi_n(x) = \begin{cases} -1 & \text{if} \quad x \le -\frac{1}{n} \\ nx & \text{if} \quad -\frac{1}{n} < x < \frac{1}{n} \\ 1 & \text{if} \quad x \ge \frac{1}{n}. \end{cases}$$

The filter $\langle\phi\rangle - \langle\phi\rangle$ is generated by the basis $\{\{\phi_m - \phi_k : m, k \ge n\} : n \in \mathbb{N}\}$. It is easy to see that for any $n \in \mathbb{N}$ we have

$$-\varphi_n \le \phi_m - \phi_k \le \varphi_n, \ m \ge n, k \ge n,$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is the sequence given in (13). Since $(\varphi_n)_{n\in\mathbb{N}}$ is a decreasing sequence with infimum equal to the constant zero function z, $(-\varphi_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ are sequences that can be associated with the filter $\langle\phi\rangle - \langle\phi\rangle$ in terms of the definition of λ_o, see (8). Hence the filter $\langle\phi\rangle - \langle\phi\rangle$ order converges to z, which implies that $\phi = (\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

On the other hand it is quite clear that this sequence is not order convergent in $C(X)$.

Example 21 shows that the convergence vector space $(C(X), \lambda_o)$ is not complete.

One of the main aims of the this paper is to construct a completion of the convergence vector space $(C(X), \lambda_o)$ as a set of functions defined on the same domain X. Since the convergence structure λ_o is defined through the partial order on $C(X)$ it is natural to consider the Dedekind order completion of $C(X)$. In [2] the Dedekind order completion of $C(X)$ was represented as a subset of the set of all Hausdorff continuous functions $\mathbb{H}(X)$ discussed in the next section. It was also shown that in the special case when X is a metric space the Dedekind order completion of $C(X)$ is exactly $\mathbb{H}(X)$. Let us note that the Dedekind order completion of a poset does not give automatically a completion with respect to any uniform convergence structure defined through the order. In fact it is shown in [12] that convergence with respect to the order topology on the Dedekind order completion of a poset does not imply convergence with respect to the order topology on the original poset. Hence the results given in the sequel with regard to the completion of $C(X)$ through Hausdorff continuous functions are highly nontrivial.

5. The set of Hausdorff continuous functions. The Hausdorff continuous functions are not unlike the usual real valued continuous functions. However, these functions may assume interval values on a certain subset of the domain. Hence the concept of Hausdorff continuity is formulated within the realm of interval valued functions. Let $\mathbb{IR}$ denote the set of all finite closed real intervals, that is,

$$\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a} \le \overline{a}, \ \underline{a}, \overline{a} \in \mathbb{R}\}.$$

Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a, while $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a. An interval a is called proper interval, if $w(a) > 0$ and point interval, if $w(a) = 0$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. We denote by $\mathbb{E}(X)$ the set of all locally bounded interval valued functions defined on the topological space X, that is,

$$\mathbb{E}(X) = \{f : X \to \mathbb{IR},\ f\text{-locally bounded}\}.$$

Since $\mathbb{R} \subseteq \mathbb{IR}$ the set $\mathbb{E}(X)$ contains the set

$$\mathcal{E}(X) = \{f : X \to \mathbb{R},\ f\text{-locally bounded}\}$$

of all locally bounded real functions defined on X.

For every $x \in X$, $\mathcal{V}_x$ denotes the set of neighborhoods of x. The pair of mappings $I, S : \mathbb{E}(X) \to \mathcal{E}(X)$ defined by

$$I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf\{z \in f(y) : y \in V\}, \tag{14}$$

$$S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup\{z \in f(y) : y \in V\}. \tag{15}$$

are called lower Baire, and upper Baire operators, respectively. Clearly for every $f \in \mathbb{E}(X)$ we have

$$I(f)(x) \leq z \leq S(f)(x),\ z \in f(x),\ x \in X. \tag{16}$$

Hence the mapping $F : \mathbb{E}(X) \to \mathbb{E}(X)$, called the graph completion operator, where

$$F(f)(x) = [I(f)(x), S(f)(x)],\ x \in X,\ f \in \mathbb{E}(X), \tag{17}$$

is well defined and we have the inclusion

$$f(x) \subseteq F(f)(x),\ x \in X. \tag{18}$$

DEFINITION 22. A function $f \in \mathbb{E}(X)$ is called Hausdorff continuous, or H-continuous, if for every $g \in \mathbb{E}(X)$ which satisfies the inclusion $g(x) \subseteq f(x)$, $x \in X$, we have $F(g)(x) = f(x)$, $x \in X$.

The concepts of Hausdorff continuity is strongly connected with the concepts of semi-continuity of real functions. We have the following characterization of the fixed points of the lower and the upper Baire operators:

$$I(f) = f \Longleftrightarrow f \text{ is lower semi-continuous on } X\ , \tag{19}$$

$$S(f) = f \Longleftrightarrow f \text{ is upper semi-continuous on } X\ . \tag{20}$$

Let $f \in \mathbb{E}(X)$. For every $x \in X$ the value of f is an interval $[\underline{f}(x), \overline{f}(x)]$. Hence, the function f can be written in the form $f = [\underline{f}, \overline{f}]$, where $\underline{f}, \overline{f} \in \mathcal{E}(X)$ and $\underline{f} \leq \overline{f}$. The lower and upper Baire operators as well as the graph completion operator of an interval valued function $f = [\underline{f}, \overline{f}] \in \mathbb{E}(X)$ can be conveniently represented in terms of the functions $\underline{f}$ and $\overline{f}$:

$$I(f) = I(\underline{f})\ ,\quad S(f) = S(\overline{f})\ ,\quad F(f) = [I(\underline{f}), S(\overline{f})].$$

We have the following characterization of the H-continuous functions, see [2]:

$$f=[\underline{f},\overline{f}] \text{ is H-continuous} \Longleftrightarrow \begin{cases} \overline{f} \text{ is upper semi-continuous} \\ \underline{f} \text{ is lower semi-continuous} \\ f=F(\underline{f})=F(\overline{f}). \end{cases} \tag{21}$$

The concept of H-continuity can be considered as a generalization of the concept of continuity of real functions in the sense that the only real (point valued) functions contained in $\mathbb{H}(X)$ are the continuous functions, that is,

$$\left.\begin{array}{c} f\in\mathcal{E}(X) \\ f \text{ is H-continuous} \end{array}\right\} \Longrightarrow f \quad \text{is continuous.} \tag{22}$$

With every function $f\in\mathbb{E}(X)$ one can associate Hausdorff continuous functions as stated in the following theorem, [2], [14].

THEOREM 23. *Let $f\in\mathbb{E}(X)$. Then both functions $F(S(I(f)))$ and $F(I(S(f)))$ are H-continuous.*

A partial order which extends the total order on $\mathbb{R}$ can be defined on $\mathbb{IR}$ in more than one way. However, it proves useful to consider on $\mathbb{IR}$ the partial order $\leq$ defined by

$$[\underline{a},\overline{a}]\leq[\underline{b},\overline{b}] \Longleftrightarrow \underline{a}\leq\underline{b},\ \overline{a}\leq\overline{b}. \tag{23}$$

The partial order induced on $\mathbb{E}(X)$ by (23) in a point-wise way, i.e.,

$$f\leq g \Longleftrightarrow f(x)\leq g(x),\ x\in X, \tag{24}$$

is an extension of the usual point-wise order on the set of real valued functions $\mathcal{E}(X)$. Note that the Baire operators and the graph completion operator are all monotone increasing with respect to the order (24), that is, for $f,g\in\mathbb{E}(X)$ we have

$$f\leq g \Longrightarrow (\ I(f)\leq I(g),\ S(f)\leq S(g),\ F(f)\leq F(g)\ ). \tag{25}$$

We denote by $\mathbb{H}(X)$ the set of all Hausdorff continuous interval valued functions defined on X. The partial order (24) induces on $\mathbb{H}(X)$ a sequential convergence structure σ_o as shown in Section 3. An important property of the set $\mathbb{H}(X)$ is that it is Dedekind order complete with respect to the partial order (24), see [2]. Using the Dedekind order completeness of $\mathbb{H}(X)$ the order convergence can be defined on $\mathbb{H}(X)$ in the following equivalent way. A sequence $(f_n)_{n\in\mathbb{N}}\subset\mathbb{H}(X)$ order converges to $f\in\mathbb{H}(X)$ if and only if it is bounded and we have

$$\begin{aligned} f &= \operatorname{liminf} f_n = \sup_{n\in\mathbb{N}}\inf\{f_m : m\geq n\}, \\ f &= \operatorname{limsup} f_n = \inf_{n\in\mathbb{N}}\sup\{f_m : m\geq n\}. \end{aligned} \tag{26}$$

We should note that the infima and suprema in the above statement are defined through the order (24) on $\mathbb{H}(X)$ and are not the point-wise ones. They all exist since the sequence is bounded and the set $\mathbb{H}(X)$ is Dedekind order complete. The following theorem gives a usefull characterization of the infima and suprema on $\mathbb{H}(X)$ in terms of the point-wise ones.

THEOREM 24. *Let $\mathcal{F}$ be a bounded subset of $\mathbb{H}(\Omega)$ and let the functions $\varphi, \psi \in \mathcal{E}(X)$ be defined by*

$$\varphi(x) = \inf\{\underline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{F}\}\ ,\ \ \psi(x) = \sup\{\overline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{F}\}\ ,\ x \in X.$$

Then

$$\inf \mathcal{F} = F(I(\varphi)),\quad \sup \mathcal{F} = F(S(\psi)).$$

Furthermore if the set $\mathcal{F}$ is finite then

$$\inf \mathcal{F} = F(\varphi),\quad \sup \mathcal{F} = F(\psi).$$

The proof is given in [2] [Proof of Theorem 5].

The set $\mathbb{H}(X)$ is a lattice with respect to the partial order given in (24). Hence $(\mathbb{H}(X), \sigma_o)$, where σ_o is given in (4), is a sequential convergence space. However, since $\mathbb{H}(X)$ is not a vector lattice, to show that $(\mathbb{H}(X), \sigma_o)$ is an FS-space we cannot use the same approach as for the space $(C(X), \sigma_o)$. The next theorem shows that, under some very general assumption for X, $(\mathbb{H}(X), \sigma_o)$ is an FS-space.

THEOREM 25. *For any completely regular topological space X the sequential convergence space $(\mathbb{H}(X), \sigma_o)$ is an FS-space.*

The proof is similar to the proof of the corresponding statement in Theorem 17, this time using Lemma 37 instead of the respective result in [9].

From the above theorem and Theorem 7 it follows that, if the topological space X is completely regular, the sequential convergence structure σ_o on $\mathbb{H}(X)$ is induced by a convergence structure on $\mathbb{H}(X)$. Furthermore, using also Theorem 16 we obtain that the mapping λ_o given in (8) is a convergence structure on $\mathbb{H}(X)$ which induces σ_o. The convergence space $(\mathbb{H}(X), \lambda_o)$ will play an important role in the construction of the convergence vector space completion of $(C(X), \lambda_o)$.

6. Convergence vector space completion of $C(X)$. In the two convergence spaces $(C(X), \lambda_o)$ and $(\mathbb{H}(X), \lambda_o)$ the meaning of the symbol λ_o is different and needs to be considered in the respective context, e.g. in $(C(X), \lambda_o)$ the domain of the mapping denoted by λ_o is $C(X)$ while in $(\mathbb{H}(X), \lambda_o)$ the domain of the mapping denoted by λ_o is $\mathbb{H}(X)$. Since in this section we will consider an interplay between the two convergence spaces, to avoid possible confusion we will denote the order convergence structure on $C(X)$ by λ_c and the order convergence structure on $\mathbb{H}(X)$ by λ_h. Similarly, for the respective sequential order convergence structures we will use the notations σ_c for the sequential order convergence structure on $C(X)$ and σ_h for the sequential order convergence structure on $\mathbb{H}(X)$. The suprema and infima in this section are all considered in the set $\mathbb{H}(X)$. Since $C(X)$ is a sublattice of $\mathbb{H}(X)$, the infimum or supremum in $C(X)$ of a subset of $C(X)$, if exists, coincides with the respective infimum or supremum in $\mathbb{H}(X)$.

Our aim is to show that the convergence space $(\mathbb{H}(X), \lambda_h)$ is the completion of the convergence vector space $(C(X), \lambda_c)$. For the underlying sets we have the inclusion

$$C(X) \subseteq \mathbb{H}(X).$$

The first question is if the convergence structure λ_h is an extension of the convergence structure λ_c, or equivalently, if λ_c is the subspace convergence structure induced by λ_h on $C(X)$. In addressing this issue the next theorem is instrumental. Since this theorem holds only when X is a metric space, the completion of $C(X)$ will be characterized in this section under the additional assumption of X being a metric space.

THEOREM 26. *Let X be a metric space. Then for every function $f \in \mathbb{H}(X)$ there exists an increasing sequence $(\xi_n)_{n\in\mathbb{N}}$ on $C(X)$ and a decreasing sequence $(\eta_n)_{n\in\mathbb{N}}$ on $C(X)$ such that*

$$f = \sup_{n\in\mathbb{N}} \xi_n = \inf_{n\in\mathbb{N}} \eta_n.$$

The proof is based on a construction used in proving similar result for lower and upper semi-continuous functions and also used in [2] [Theorem 11]. For completeness of the exposition the proof is given in the Appendix. Using Theorem 26 we prove the following result.

THEOREM 27. *Let X be a metric space.*
a) For every increasing sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$ on $\mathbb{H}(X)$ which is bounded from above there exists an increasing sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ on $C(X)$ such that

$$\alpha_n \le \xi_n,\ n \in \mathbb{N}, \quad \text{and} \quad \sup_{n\in\mathbb{N}} \xi_n = \sup_{n\in\mathbb{N}} \alpha_n.$$

b) For every decreasing sequence $\eta = (\eta_n)_{n\in\mathbb{N}}$ on $\mathbb{H}(X)$ which is bounded from below there exists a decreasing sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ on $C(X)$ such that

$$\beta_n \ge \eta_n,\ n \in \mathbb{N}, \quad \text{and} \quad \inf_{n\in\mathbb{N}} \eta_n = \inf_{n\in\mathbb{N}} \beta_n.$$

Proof. We will only give the proof of a) because b) is proved in a similar way. It follows from Theorem 26 that for every $n \in \mathbb{N}$ there exist an increasing sequence $(p_{nm})_{m\in\mathbb{N}}$ on $C(X)$ such that

$$\xi_n = \sup_{m\in\mathbb{N}} p_{nm}\ .$$

Denote

$$\alpha_n = \sup\{p_{1n}, p_{2n}, ..., p_{nn}\}\ ,\ n \in \mathbb{N}\ .$$

For every $n \in \mathbb{N}$, α_n is a supremum of a finite set of continuous functions. Hence it equals their point-wise supremum and is a continuous function, that is, $\alpha_n \in C(X)$. Using Lemma 36 we obtain that $(\alpha_n)_{n\in\mathbb{N}}$ is an increasing sequence and $\sup_{n\in\mathbb{N}} \xi_n = \sup_{n\in\mathbb{N}} \alpha_n$. □

Denote by $e : C(X) \to \mathbb{H}(\Omega)$ the inclusion mapping, that is, $e(f) = f$, $f \in C(X)$. The subspace convergence structure induced by λ_h on $C(X)$ is defined as follows:

A filter $\mathcal{A}$ on $C(X)$ converges to $f \in C(X)$ if and only if $e(\mathcal{A}) \in \lambda_h(f)$.

Note that the image filter $e(A)$ is actually a filter on $\mathbb{H}(\Omega)$ which is generated by $\mathcal{A}$ considered as a subset of the power set of $\mathbb{H}(\Omega)$.

THEOREM 28. *Let X be a metric space. Then, the convergence structure λ_c is the subspace convergence structure induced by λ_h on $C(X)$, that is, for any filter $\mathcal{A}$ on $C(X)$ and $f \in C(X)$*

$$\mathcal{A} \in \lambda_c(f) \iff e(\mathcal{A}) \in \lambda_h(f).$$

Proof. The implication to the right follows directly from the definitions of λ_c and λ_h, see (8). We will prove the implication to the left. Let the filter $\mathcal{A}$ on $C(X)$ and $f \in C(X)$ be such that $e(\mathcal{A}) \in \lambda_h(f)$. Then there exist $(\tilde{\alpha}_n)_{n\in\mathbb{N}}$ an increasing sequence on $\mathbb{H}(X)$ and $(\tilde{\beta}_n)_{n\in\mathbb{N}}$ a decreasing sequence on $\mathbb{H}(X)$ with $f = \sup_{n\in\mathbb{N}} \tilde{\alpha}_n = \inf_{n\in\mathbb{N}} \tilde{\beta}_n$ which can be associated with $e(\mathcal{A})$ in terms of (8). Let us note that the sets represented through the interval notation in (8) depend on the poset P. Since here we consider both $P = C(X)$ and $P = \mathbb{H}(X)$ to avoid possible confusion we will write the respective sets in an explicit form. In terms of (8) the filter $\mathcal{B}$ on $\mathbb{H}(X)$ is generated by a basis of the form $\{\{\phi \in \mathbb{H}(X) : \tilde{\alpha}_n \leq \phi \leq \tilde{\beta}_n\} : n \in \mathbb{N}\}$ and is coarser then $e(\mathcal{A})$.

Using Theorem 27 there exist sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ on $C(X)$ such that

$$\begin{aligned} &\alpha_n \leq \tilde{\alpha}_n \quad \beta_n \geq \tilde{\beta}_n \quad n \in \mathbb{N} \\ &\sup_{n\in\mathbb{N}} \alpha_n = \inf_{n\in\mathbb{N}} \beta_n = f. \end{aligned} \tag{27}$$

We will show that the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ can be associated with the filter $\mathcal{A}$ and the function f in terms of the definition of λ_c, see (8). To this end it remains to prove that the filter generated by the basis

$$\{\{\phi \in C(X) : \alpha_n \leq \phi \leq \beta_n\} : n \in \mathbb{N}\} \tag{28}$$

is coarser then the filter $\mathcal{A}$. It is sufficient to show that every set in the basis given in (28) above is a superset of a set in $\mathcal{A}$. Since the filter $e(\mathcal{A})$ is generated by $\mathcal{A}$ and using that the filter $\mathcal{B}$ is coarser than $e(\mathcal{A})$, for every $n \in \mathbb{N}$ there exists a set $A \in \mathcal{A}$ such that

$$A \subset \{\phi \in \mathbb{H}(X) : \tilde{\alpha}_n \leq \phi \leq \tilde{\beta}_n\}.$$

Using that A is a subset of $C(X)$, and the inequalities (27) we obtain

$$A \subset \{\phi \in C(X) : \tilde{\alpha}_n \leq \phi \leq \tilde{\beta}_n\} \subset \{\phi \in C(X) : \alpha_n \leq \phi \leq \beta_n\}.$$

Hence the filter generated by the basis (28) is coarser than $\mathcal{A}$.

Now all conditions in the definition of λ_c as given in (8) with regard to the filter $\mathcal{A}$ and the function f are satisfied which implies that $\mathcal{A} \in \lambda_c(f)$. □

The next theorem shows that the Cauchy sequences on $(C(X), \lambda_c)$ converge on $(\mathbb{H}(X), \lambda_h)$.

THEOREM 29. *For every Cauchy sequence ξ on the convergence vector space $(C(X),\lambda_c)$ there exists $f\in\mathbb{H}(X)$ such that $\xi\in\lambda_h(f)$.*

Proof. Let ξ be a Cauchy sequence. According to Theorem 18 there exist a decreasing sequence $\beta=(\beta_n)_{n\in\mathbb{N}}$ on $C(X)$ such that

$$\xi_m-\xi_k\le\beta_n,\ m,k\ge n,\ n\in\mathbb{N}, \tag{29}$$

$$z=\sup_{n\in\mathbb{N}}\alpha_n=\inf_{n\in\mathbb{N}}\beta_n, \tag{30}$$

where z is the constant zero function. Since a Cauchy sequence is order bounded and $\mathbb{H}(X)$ is Dedekind order complete we can define the Haudorff continuous functions

$$f=[\underline{f},\overline{f}]=\operatorname{liminf}\xi\ ,\quad g=[\underline{g},\overline{g}]=\operatorname{limsup}\xi\ .$$

To prove the theorem we need to show that $f=g$, see (26). From the inequality (29) we have

$$\xi_m\le\xi_k+\beta_n,\ m,k\ge n,\ n\in\mathbb{N}.$$

Going with m to infinity we obtain

$$g=\operatorname{limsup}\xi\le\xi_k+\beta_n,\ k\ge n,\ n\in\mathbb{N}. \tag{31}$$

It is easy to see that the interval function

$$g-\beta_n=[\underline{g}-\beta_n,\overline{g}-\beta_n]$$

is Hausdorff continuous and that the inequality (31) implies

$$g-\beta_n\le\xi_k,\ k\ge n,\ n\in\mathbb{N}.$$

Taking limit inferior as $k\to\infty$ we obtain

$$g-\beta_n\le\operatorname{liminf}\xi=f,\ n\in\mathbb{N},$$

which also implies

$$\underline{g}-\overline{f}\le\beta_n,\ n\in\mathbb{N}.$$

Using the monotonicity of the operators S and F and that $\beta_n\in C(X)$ we have

$$F(S(\underline{g}-\overline{f}))\le F(S(\beta_n))=\beta_n,\ n\in\mathbb{N}.$$

Since $F(S(\underline{f}-\overline{g}))$ is a Hausdorff continuous function, see Theorem 23 and (19), which is a lower bound of $(\beta_n)_{n\in\mathbb{N}}$ then

$$F(S(\underline{g}-\overline{f}))\le\inf_{n\in\mathbb{N}}\beta_n=z$$

or equivalently

$$F(S(\underline{g}-\overline{f}))(x)\le 0\ ,\quad x\in X.$$

Therefore

$$\underline{g}(x)-\overline{f}(x)\le S(\underline{g}-\overline{f})(x)\le 0\ ,\quad x\in X\ ,$$

which implies

$$\underline{g} \leq \overline{f}.$$

From the above inequality using (21) and the monotonicity of F we obtain

$$g = F(\underline{g}) \leq F(\overline{f}) = f.$$

Since the inequality $f \leq g$ is obvious the above implies that $f = g$ which completes the proof. □

We show next that $C(X)$ is dense in $\mathbb{H}(X)$.

THEOREM 30. *For every function $f \in \mathbb{H}(X)$ there exists a Cauchy sequence ξ on the convergence vector space $(C(X), \lambda_c)$ such that $\xi \to f$ in the convergence space $(\mathbb{H}(X), \lambda_h)$.*

Proof. Let $f \in \mathbb{H}(X)$. According to Theorem 26 there exists an increasing sequence ξ on $C(X)$ such that $\sup_{n\in\mathbb{N}} \xi_n = f$. It is easy to see that ξ order converges to f. Indeed, for the two sequences α and β required in Definition 14 we can take $\alpha = \xi$ and β the constant sequence with value f.

To complete the proof it remains to show that the sequence ξ is a Cauchy sequence in the space $(C(X), \lambda_c)$. It follows from Theorem 26 that there exists a decreasing sequence $\eta = (\eta_n)_{n\in\mathbb{N}}$ such that $f = \inf_{n\in\mathbb{N}} \eta_n$. Clearly the sequence $(\eta_n - \xi_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below by the constant zero function z. Furthermore, we can see that

$$\inf_{n\in\mathbb{N}} (\eta_n - \xi_n) = z.$$

Indeed, assume that $\phi \in C(X)$ is a lower bound of $(\eta_n - \xi_n)_{n\in\mathbb{N}}$. Then we have

$$\beta_m \geq \xi_k + \phi \ , \quad m, k \in \mathbb{N},$$

which implies

$$f = \inf_{m\in\mathbb{N}} \eta_m \geq \xi_k + \phi \ , \quad k \in \mathbb{N}.$$

Using that the function $f - \phi = [\underline{f} - \phi, \overline{f} - \phi]$ is Hausdorff continuous and the inequality

$$f - \phi = [\underline{f} - \phi, \overline{f} - \phi] \geq \underline{f} - \phi \geq \xi_k \ , \quad k \in \mathbb{N},$$

we obtain

$$[\underline{f} - \phi, \overline{f} - \phi] \geq \sup_{k\in\mathbb{N}} \xi_k = f = [\underline{f}, \overline{f}].$$

It follows from the above inequality that $\phi \leq z$. Therefore z is the infimum of the sequence $(\eta_n - \xi_n)_{n\in\mathbb{N}}$.

We also have

$$\xi_m - \xi_k \leq \eta_m - \xi_k \leq \eta_n - \xi_n \ , \quad m, k \geq n, \ n \in \mathbb{N}.$$

Then the fact that ξ is a Cauchy sequence follows from Theorem 18. $\square$

In order to show that $\mathbb{H}(X)$ is a convergence space completion of $C(X)$ we need to introduce on $\mathbb{H}(X)$ the operations of a vector space and show that $\mathbb{H}(X)$ is a complete convergence vector space with regard to these operations and the convergence structure λ_h. We will define on $\mathbb{H}(X)$ the operations addition and scalar multiplication by extending the respective operations on $C(X)$ using that $C(X)$ is dense in $\mathbb{H}(X)$, see Theorem 30.

DEFINITION 31. Let $f, g, h \in \mathbb{H}(X)$. We say that $h = f+g$ if there exists sequences ξ and η on $C(X)$ such that $\xi \to f$, $\eta \to g$ and $\xi + \eta \to h$.

The fact that the addition given above is defined for every two functions $f, g \in \mathbb{H}(X)$ follows directly from Theorems 29 and 30. Indeed, according to Theorem 30, for any functions $f, g \in \mathbb{H}(X)$ there exists sequences ξ and η such that $\xi \to f$, $\eta \to g$. Using that ξ and η are Cauchy it is easy to see that $\xi + \eta$ is also Cauchy. Then it follows from Theorem 29 that there exists $h \in \mathbb{H}(X)$ such that $\xi + \eta \to h$.

We also need to show that the definition of addition does not depend on a particular choice of the sequences ξ and η. Let us assume that the sequences $\xi^{(i)}, \eta^{(i)}, i = 1, 2$, are such that $\xi^{(i)} \to f$, $\eta^{(i)} \to g$, $i = 1, 2$. We will show that $\xi^{(1)} + \eta^{(1)}$ and $\xi^{(2)} + \eta^{(2)}$ converge to the same limit. Consider the trivial mixtures $\xi = \xi^{(1)} \diamond \xi^{(2)}$ and $\eta = \eta^{(1)} \diamond \eta^{(2)}$. Since $\mathbb{H}(X)$ is an FS-space, see Theorem 25, we have $\xi \to f$ and $\eta \to g$. Clearly the sequences $\xi^{(1)} + \eta^{(1)}$ and $\xi^{(2)} + \eta^{(2)}$ are both subsequences of the Cauchy sequence $\xi + \eta$. Hence they converge to the same limit, namely, the limit of $\xi + \eta$.

The addition on $\mathbb{H}(X)$ can be characterized in the following equivalent way

$$h = f + g \Longleftrightarrow \begin{cases} \text{For every two sequences } \xi \text{ and } \eta \text{ on C(X),} \\ \text{where } \xi \to f,\ \eta \to g, \text{ we have } \xi + \eta \to h. \end{cases} \tag{32}$$

It is important to mention here that the operation addition given in Definition 31 is different from the point-wise addition of Hausdorff continuous interval functions, since it is well known that point-wise sum of $f, g \in \mathbb{H}(X)$ using the usual addition of intervals, see [1], is not necessarily a Hausdorff continuous function. This is also illustrated by the following example.

EXAMPLE 32. Consider the functions $f, g \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ [0,1], & \text{if } x = 0, \\ 1, & \text{if } x > 0; \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x < 0, \\ [-1,0], & \text{if } x = 0, \\ -1, & \text{if } x > 0. \end{cases}$$

Using addition of intervals we have

$$f(x)+g(x)=\begin{cases} 0, & \text{if} \quad x<0,\\ [-1,1], & \text{if} \quad x=0,\\ 0, & \text{if} \quad x>0. \end{cases}$$

We can obtain the sum with respect to the addition given in Definition 31 by using the following sequence of continuous functions:

$$\xi_n(x)=\begin{cases} 0, & \text{if} \quad x\le 0,\\ nx, & \text{if} \quad 0<x\le \frac{1}{n}\\ 1, & \text{if} \quad x>\frac{1}{n}. \end{cases}$$

It is easy to see that $\xi \to f$ while $-\xi \to g$. Therefore $f+g$, being the limit of $\xi-\xi$, is the constant zero function.

The operation of scalar multiplication is defined as follows.

DEFINITION 33. Let $f, g \in \mathbb{H}(X)$ and $a \in \mathbb{R}$. Then we say that $g = af$ if there exist a sequence ξ on $C(X)$ such that $\xi \to f$ and $a\xi \to g$.

Similar to the operation addition we show that scalar multiplication is defined for all $f \in \mathbb{H}(X)$ and all $a \in \mathbb{R}$. However, in this case the operation coincides exactly with the respective point-wise operation. More precisely, for any $f = [\underline{f}, \overline{f}]$ and $a \in \mathbb{R}$ we have

$$(af)(x)=a(f(x))=\begin{cases} [a\underline{f}(x), a\overline{f}(x)], & \text{if} \quad a\ge 0\\ [a\overline{f}(x), a\underline{f}(x)], & \text{if} \quad a<0. \end{cases}$$

THEOREM 34. *The convergence space $(\mathbb{H}(X), \lambda_h)$ is a complete convergence vector space with respect to the operations given in Definitions 31 and 33.*

Proof. First we need to show that $\mathbb{H}(X)$ is a vector lattice with respect to the given operations and the partial order $\le$ given in (24). However, the verification of the respective axioms uses standard techniques and will be omitted. It is also an easy exercise to see that the vector lattice $\mathbb{H}(X)$ is Archimedean. We proceed by showing that $(\mathbb{H}(X), \lambda_h)$ is a convergence vector space. To this end we apply Theorem 17. Since $(\mathbb{H}(X), \sigma_h)$ is an Archimedean vector lattice it follows that $(\mathbb{H}(X), \lambda_h)$ is a convergence vector space. Finally we will show that the convergence vector space $(\mathbb{H}(X), \lambda_h)$ is complete by using the Dedekind order completeness of $\mathbb{H}(X)$. Since the convergence space $(\mathbb{H}(X), \lambda_h)$ is first countable sequential argument is sufficient with regard to its completeness, see [5] [Proposition 3.6.5]. Let $\xi = (\xi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence on $\mathbb{H}(X)$. According to Theorem 18 there exists a decreasing sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ with $\inf\limits_{n\in\mathbb{N}} \beta_n = z$, z being the constant zero function, such that

$$\xi_m - \xi_k \le \beta_n, \ \ m, k \ge n, \ \ n \in \mathbb{N}.$$

Since m and k can vary independently the above inequality yields

$$\operatorname{limsup} \xi - \operatorname{liminf} \xi \leq \beta_n, \ n \in \mathbb{N}.$$

Therefore

$$\operatorname{limsup} \xi - \operatorname{liminf} \xi \leq \inf_{n \in \mathbb{N}} \beta_n = z,$$

or, equivalently,

$$\operatorname{limsup} \xi = \operatorname{liminf} \xi.$$

Let $f = \operatorname{limsup} \xi = \operatorname{liminf} \xi$. Then the sequence ξ and the function f satisfy the condition (26) which implies that ξ order converges to f. Hence $(\mathbb{H}(X), \lambda_h)$ is a complete convergence vector space. □

Summarizing the results presented in this section we can say that $(\mathbb{H}(X), \lambda_h)$ is a convergence vector space completion of $(C(X), \lambda_c)$ in the following very direct sense

- $(\mathbb{H}(X), \lambda_h)$ is a complete convergence vector space, see Theorem 34;
- $(C(X), \lambda_c)$ is a convergence vector subspace of $(\mathbb{H}(X), \lambda_h)$ meaning that both the convergence structure and the operations on $C(X)$ are the ones induced by $(\mathbb{H}(X), \lambda_h)$, see Theorem 28 and Definitions 31, 33;
- $C(X)$ is dense in $\mathbb{H}(X)$, see Theorem 30, so that no proper complete convergence subspace of $(\mathbb{H}(X), \lambda_h)$ contains $C(X)$.

REMARK 35. The operations addition and scalar multiplication given in Definitions 31 and 33 introduced on $\mathbb{H}(X)$ the structure of a vector space. The definitions of these operations were formulated in order to satisfy a particular need, namely, continuous extension of the respective operations on $C(X)$. It is interesting therefore that these operations coincide with the operations addition and scalar multiplication defined in [3] in a completely different way for the case when X is an open subset of $\mathbb{R}^d$. It was shown in [3] that, under certain assumptions, this is the only way to define the vector space operations on $\mathbb{H}(X)$ and that $\mathbb{H}(X)$ is the largest vector space of interval functions.

7. Conclusion.

This paper brings together three concepts which have not been related so far, namely, the concept of order convergence, the concept of convergence space and the concept of Hausdorff continuous functions. The general result that the order convergence on any vector lattice is induced by a convergence structure given in Section 3 is particularly significant in view of the fact that the order convergence is typically not induced by a topology. The primary focus is on the vector lattice $C(X)$. The order convergence on $C(X)$ is indeed not topological as it violates the Urysohn property and the diagonal property which are associated with a topologically induced convergence, thus justifying our interest in the convergence vector space $(C(X), \lambda_o)$, where the convergence structure λ_o induces the order convergence. A uniform structure, including the concept of Cauchy sequences, is induced in a natural way through the vector space operations on any convergence vector space, in particular on $(C(X), \lambda_o)$ as well. However, the convergence vector space $(C(X), \lambda_o)$ is not complete. The main result in this regard is that the completion of $(C(X), \lambda_o)$ can be given through the larger set of Hausdorff continuous functions $\mathbb{H}(X)$. Within this construction we define operations on $\mathbb{H}(X)$ which induce on $\mathbb{H}(X)$ the structure of a vector space, a result which is quite remarkable particularly in view of the fact that no set of interval functions which includes proper interval functions is a vector space with respect to the point-wise defined operations. The convergence vector space $(\mathbb{H}(X), \lambda_o)$ may have many applications due to its completeness. An application of the set $\mathbb{H}(X)$ to the solution of nonlinear PDEs through the order completion method which uses its Dedekind completeness was reported in [4]. Further research will seek applications to the solution of PDE by using the order convergence structure on $\mathbb{H}(X)$.

Appendix. This appendix contains some technical results used in the main body of the paper. The lemma below is useful as a substitute for the diagonal property of the sequential convergence which does not hold in general for the sequential order convergence structure σ_o on a poset.

LEMMA 36. *Let P be lattice with respect to a given partial order $\leq$.*

a) Let for every $n \in \mathbb{N}$ the sequence $(p_{nm})_{m\in\mathbb{N}}$ on P be increasing and let

$$\xi_n = \sup_{m\in\mathbb{N}} p_{nm},\ n \in \mathbb{N},$$

exist. If the sequence $(\xi_n)_{n\in\mathbb{N}}$ is increasing and $\sup_{n\in\mathbb{N}} \xi_n$ exists then the sequence $(\alpha_n)_{n\in\mathbb{N}}$, where

$$\alpha_n = \sup\{p_{1n}, p_{2n}, ..., p_{nn}\},\ n \in \mathbb{N},$$

is also increasing, $\sup_{n\in\mathbb{N}} \alpha_n$ exists and $\sup_{n\in\mathbb{N}} \alpha_n = \sup_{n\in\mathbb{N}} \xi_n$.

b) Let for every $n \in \mathbb{N}$ the sequence $(q_{nm})_{m\in\mathbb{N}}$ on P be decreasing and let

$$\eta_n = \inf_{m\in\mathbb{N}} q_{nm},\ n \in \mathbb{N},$$

exist. If the sequence $(\eta_n)_{n\in\mathbb{N}}$ is decreasing and $\sup_{n\in\mathbb{N}} \eta_n$ exists then the sequence $(\beta_n)_{n\in\mathbb{N}}$ where

$$\beta_n = \inf\{q_{1n}, q_{2n}, ..., q_{nn}\},\ n \in \mathbb{N},$$

is also decreasing, $\inf_{n\in\mathbb{N}} \beta_n$ exists and $\inf_{n\in\mathbb{N}} \beta_n = \inf_{n\in\mathbb{N}} \eta_n$.

Proof. a) Let the sequence $(\xi_n)_{n\in\mathbb{N}}$ be increasing and let $f = \sup_{n\in\mathbb{N}} \xi_n$. Using the monotonicity of the sequences $(p_{nm})_{m\in\mathbb{N}}$, $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha_n &= \sup\{p_{1n}, p_{2n}, ..., p_{nn}\} \leq \sup\{p_{1\,n+1}, p_{2\,n+1}, ..., p_{n\,n+1}\} \\ &\leq \sup\{p_{1\,n+1}, p_{2\,n+1}, ..., p_{n\,n+1}, p_{n+1\,n+1}\} = \alpha_{n+1} \end{aligned}$$

which implies that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is increasing. We will show next that $(\alpha_n)_{n\in\mathbb{N}}$ is bounded and $f = \sup\limits_{n\in\mathbb{N}} \alpha_n$. Since for $k \leq n$ we have $p_{kn} \leq \xi_k \leq \xi_n$ then

$$\alpha_n \leq \xi_n \leq f \ , \ n \in \mathbb{N}. \tag{33}$$

Hence f is and upper bound of $(\alpha_n)_{n\in\mathbb{N}}$. Let g be any upper bound of $(\alpha_n)_{n\in\mathbb{N}}$. We will show that $f \leq g$ which implies that $f = \sup\limits_{n\in\mathbb{N}} \alpha_n$. It is easy to see that $p_{km} \leq g$, $k, m \in \mathbb{N}$. Indeed,

if $k \leq m$ then $p_{km} \leq \alpha_m \leq g$;

if $k > m$ then $p_{km} \leq p_{kk} \leq \alpha_k \leq g$.

Therefore

$$\xi_k = \sup_{m\in\mathbb{N}} p_{km} \leq g, \ k \in \mathbb{N}.$$

Hence

$$f = \sup_{k\in\mathbb{N}} \xi_k \leq g$$

which implies $f = \sup\limits_{n\in\mathbb{N}} \alpha_n$.

b) is proved in a similar way. □

The lemma below is used in the proof of Theorem 25. Note that all infima and suprema are in terms of the partial order (24) on $\mathbb{H}(X)$ and are generally different from the point-wise ones.

LEMMA 37. *a) Let* $\left(\alpha_n^{(1)}\right)_{n\in\mathbb{N}}$ *and* $\left(\alpha_n^{(2)}\right)_{n\in\mathbb{N}}$ *be increasing sequences on* $\mathbb{H}(X)$ *and* $f \in \mathbb{H}(X)$. *If*

$$f = \sup_{n\in\mathbb{N}} \alpha_n^{(1)} = \sup_{n\in\mathbb{N}} \alpha_n^{(2)}$$

then

$$f = \sup_{n\in\mathbb{N}} \inf\{\alpha_n^{(1)}, \alpha_n^{(2)}\}.$$

b) Let $\left(\beta_n^{(1)}\right)_{n\in\mathbb{N}}$ *and* $\left(\beta_n^{(2)}\right)_{n\in\mathbb{N}}$ *be decreasing sequences on* $\mathbb{H}(X)$ *and* $f \in \mathbb{H}(X)$. *If*

$$f = \inf_{n\in\mathbb{N}} \beta_n^{(1)} = \inf_{n\in\mathbb{N}} \beta_n^{(2)}$$

then

$$f = \inf_{n\in\mathbb{N}} \sup\{\beta_n^{(1)}, \beta_n^{(2)}\}.$$

Proof. We will only proof a) since b) is proved in a similar way. Denote $\alpha_n = \inf\{\alpha_n^{(1)}, \alpha_n^{(2)}\}$, $n \in \mathbb{N}$. Assume the opposite, that is, $\sup_{n\in\mathbb{N}} \alpha_n \neq f$. This means that there exists an upper bound $\phi = [\underline{\phi}, \overline{\phi}] \in \mathbb{H}(X)$ of the set $\{\alpha_n : n \in \mathbb{N}\}$ such that the inequality $\phi \geq f$ is false. Without loss of generality we may assume that $\phi \leq f$ because if this is not the case we can use $\inf\{\phi, f\}$ in the place of ϕ. Since $\phi \neq f$ there exists $a \in X$ such that $\overline{\phi}(a) < \underline{f}(a)$. Indeed, if $\overline{\phi}(x) \geq \underline{f}(x)$ for every $x \in X$ then using (21) and the monotonicity of the operator F, see (25) we have

$$\phi = F(\overline{\phi}) \geq F(\underline{f}) = f,$$

which is a contradiction since the above inequality is assumed false. Using that the function $\underline{f} - \overline{\phi}$ is lower semi continuous on X there exists an open neighborhood V of a such that

$$\underline{f}(x) - \overline{\phi}(x) > \varepsilon,\ x \in V,$$

where $\varepsilon = \frac{1}{2}(\underline{f}(a) - \overline{\phi}(a)) > 0$. Using the usual interval notations for every $n \in \mathbb{N}$ we have $\alpha_n^{(1)} = [\underline{\alpha}_n^{(1)}, \overline{\alpha}_n^{(1)}]$ and $\alpha_n^{(2)} = [\underline{\alpha}_n^{(2)}, \overline{\alpha}_n^{(2)}]$. Denote

$$M^{(1)} = \{x \in X : \underline{\alpha}_n^{(1)}(x) - \overline{\phi}(x) \leq 0,\ n \in \mathbb{N}\}$$
$$M^{(2)} = \{x \in X : \underline{\alpha}_n^{(2)}(x) - \overline{\phi}(x) \leq 0,\ n \in \mathbb{N}\}.$$

Due to the fact that the functions involved in the inequalities defining the sets $M^{(1)}$ and $M^{(2)}$ are both lower semi-continuous on X both sets are closed in the topology of X. It is easy to see that $X = M^{(1)} \bigcup M^{(2)}$. Indeed, if $x \notin M^{(1)} \bigcup M^{(2)}$ there exist m_1 and m_2 such that $\underline{\alpha}_{m_1}^{(1)}(x) > \overline{\phi}(x)$ and $\underline{\alpha}_{m_2}^{(2)}(x) > \overline{\phi}(x)$. Let $m \geq \max\{m_1, m_2\}$. Then

$$\underline{\alpha}_m^{(1)}(x) \geq \underline{\alpha}_{m_1}^{(1)}(x) > \overline{\phi}(x)$$
$$\underline{\alpha}_m^{(2)}(x) \geq \underline{\alpha}_{m_2}^{(2)}(x) > \overline{\phi}(x).$$

It follows from Theorem 24 that the function $\underline{\alpha}_m$ is a point-wise infimum of the functions $\underline{\alpha}_m^{(1)}$ and $\underline{\alpha}_m^{(2)}$. Hence we have

$$\underline{\alpha}_m(x) = \inf\{\underline{\alpha}_m^{(1)}, \underline{\alpha}_m^{(2)}\} > \overline{\phi}(x),$$

which is a contradiction because ϕ is an upper bound of α_m. Therefore $X = M^{(1)} \bigcup M^{(2)}$.

We shall see next that at least one of the sets $V \bigcap M^{(1)}$ and $V \bigcap M^{(2)}$ has an interior point. Let $V \bigcap M^{(1)}$ have no interior points. This means that every point $y \in V \bigcap M^{(1)}$ is an accumulation point of $M^{(2)}$. Using that $M^{(2)}$ is closed, this implies $y \in M^{(2)}$. Hence $V \bigcap M^{(1)} \subseteq M^{(2)}$. Therefore

$$V = (V \bigcap M^{(1)}) \bigcup (V \bigcap M^{(2)}) \subseteq M^{(1)} \bigcup M^{(2)} = M^{(2)},$$

where the set V, being an open set, certainly contains interior points.

Let b be an interior point of $V \bigcap M^{(l)}$ where $l = 1$ or $l = 2$. There exists an open neighborhood U of b such that $U \subseteq V \bigcap M^{(l)}$. Due to the complete regularity of X there exists a continuous function ψ on X such that

$$\begin{aligned}&\psi(b) = 1\\ &0 \leq \psi(x) \leq 1, \ x \in U\\ &\psi(x) = 0, \ x \notin U.\end{aligned}$$

Now the function $g = f - \varepsilon\psi \in \mathbb{H}(X)$ is an upper bound of the sequence $(\alpha_n^{(l)})_{n\in\mathbb{N}}$. Indeed, for $x \in U$ and using that $U \subseteq V \bigcap M^{(l)}$ we have

$$\underline{g} = \underline{f}(x) - \varepsilon\psi(x) \geq \underline{f}(x) - \varepsilon > \overline{\phi}(x) \geq \underline{\alpha}_n^{(l)}(x), \ n \in \mathbb{N}.$$

For $x \notin U$,

$$\underline{g} = \underline{f}(x) - \varepsilon\psi(x) = \underline{f}(x) \geq \underline{\alpha}_n^{(l)}(x), \ n \in \mathbb{N}.$$

Hence, using (21) and the monotonicity of the operator F, see (25), we obtain

$$g = F(\underline{g}) \geq F(\underline{\alpha}_n^{(l)}) = \alpha_n^{(l)},$$

which implies that $g = f - \varepsilon\psi$ is an upper bound of $(\alpha_n^{(l)})_{n\in\mathbb{N}}$. However the inequality $g(x) \geq f(x)$ is false at least at $x = b$. This is a contradiction with the fact that $f = \sup\limits_{n\in\mathbb{N}} \alpha_n^{(l)}$ and shows that $\sup\limits_{n\in\mathbb{N}} \alpha_n = f$. □

Proof of Theorem 26. We will prove the existence of an increasing sequence since the existence of a decreasing one is proved in a similar way. Let ρ be the metric on X. We will use the function $h : \mathbb{R} \to (-1, 1) \subset R$ defined by

$$h(z) = \frac{z}{1 + |z|}, \ z \in \mathbb{R}.$$

This real function is continuous and strictly increasing. The inverse function $h^{-1} : (-1, 1) \to \mathbb{R}$ is given by

$$h^{-1}(z) = \frac{z}{1 - |z|}, \ z \in (-1, 1),$$

and is also continuous and strictly increasing.

Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}(X)$. Consider the functions $\varphi_n : X \times X \to R$ defined by

$$\varphi_n(t, x) = h(\underline{f}(t)) + n\rho(t, x) = \frac{\underline{f}(t)}{1 + |\underline{f}(t)|} + n\rho(t, x), \ n \in \mathbb{N}. \tag{34}$$

It is easy to see that the function φ_n is bounded from below. Indeed, since the value of the metric ρ is always nonnegative and the fraction in (34) is greater than -1 we have $\varphi_n(t, x) > -1$. Then, we can define

$$\psi_n(x) = \inf\{\varphi_n(t, x) : t \in X\}, \ n \in \mathbb{N}.$$

First we will show that for every $n \in \mathbb{N}$ the function ψ_n is continuous on X. From the triangular inequality of the metric ρ for every $x, y, t \in X$ we have

$$\rho(t,y) - \rho(x,y) \le \rho(t,x) \le \rho(t,y) + \rho(x,y).$$

Therefore

$$\varphi_n(t,y) - n\rho(x,y) \le \varphi_n(t,x) \le \varphi_n(t,y) + n\rho(x,y).$$

Taking the infimum on $t \in X$ we obtain

$$\psi_n(y) - n\rho(x,y) \le \psi_n(x) \le \psi_n(y) + n\rho(x,y).$$

Hence we have the inequality

$$|\psi_n(x) - \psi_n(y)| \le n\rho(x,y) \ , \quad x, y \in X \ ,$$

which implies that the function ψ_n is continuous on X.

Our second step is to prove that ψ_n satisfies the inequalities

$$-1 < \psi_n(x) \le \frac{\underline{f}(x)}{1 + |\underline{f}(x)|} < 1, \ x \in X. \tag{35}$$

For every $x \in X$ we have

$$\psi_n(x) = \inf\{\varphi(t,x) : t \in X\} \le \varphi(x,x) = h(\underline{f}(x)) = \frac{\underline{f}(x)}{1 + |\underline{f}(x)|}. \tag{36}$$

Furthermore, since -1 is a lower bound of $\varphi_n(t,x)$ the inequality

$$\psi_n(x) \ge -1$$

also holds. It remains to prove that $\psi_n(x) \ne -1$. Let us assume that there exists $x \in X$ such that $\psi_n(x) = -1$. Let the real number μ be such that $-1 < \mu < \frac{\underline{f}(x)}{1+|\underline{f}(x)|}$. Then we have

$$h(\underline{f}(x)) = \frac{\underline{f}(x)}{1 + |\underline{f}(x)|} > \mu > -1.$$

Using standard techniques one can easily see that the function $h \circ \underline{f}$ is lower semi-continuous. Hence there exists $\varepsilon > 0$ such that

$$\frac{\underline{f}(t)}{1 + |\underline{f}(t)|} = h(\underline{f}(t)) > \mu \ \text{ whenever } \ \rho(t,x) < \varepsilon. \tag{37}$$

Let now $\delta = \min\{n\varepsilon, \mu + 1\}$. Since $\psi_n(x)$ is defined as an infimum on $t \in X$, there exists $t_\delta \in X$ such that

$$-1 = \psi_n(x) \le \varphi_n(t_\delta, x) \le \psi_n(x) + \delta = -1 + \delta$$

or, more precisely,

$$-1 \leq \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|} + n\rho(t_\delta, x) \leq -1+\delta.$$

Using simple manipulations we obtain

$$0 \;\leq\; \rho(t_\delta, x) \;\leq\; \frac{1}{n}\left(\delta - \left(1+\frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|}\right)\right) \;<\; \frac{\delta}{n} \;\leq\; \varepsilon \tag{38}$$

$$-1 \;\leq\; \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|} \;\leq\; -1+\delta \;\leq\; \mu. \tag{39}$$

The contradiction between inequalities (38), (39) on the one side and the condition (37) on the other side show that the assumption that $\psi_n(x) = -1$ for some $x \in X$ is false. Therefore $\psi_n(x) > -1$, $x \in X$.

We will show that $(\xi_n)_{n\in\mathbb{N}}$ where

$$\xi_n(x) = h^{-1}(\psi_n(x)) = \frac{\psi_n(x)}{1-|\psi_n(x)|},\; x \in X,\; n \in \mathbb{N}, \tag{40}$$

is the required sequence. Due to inequalities (35) the function ξ_n is well defined for every $x \in X$ and $n \in \mathbb{N}$. Moreover, ξ_n is continuous on X because ψ_n is continuous on X. Using the fact that the function h^{-1} is strictly increasing on the interval $(-1, 1)$ and that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is increasing with n we obtain that $(\xi_n)_{n\in\mathbb{N}}$ is an increasing sequence. Furthermore from the middle inequality in (35) we obtain

$$\begin{aligned}\xi_n(x) = h^{-1}(\psi_n(x)) \leq h^{-1}&\left(\frac{\underline{f}(x)}{1+|\underline{f}(x)|}\right)\\ &= h^{-1}\left(h(\underline{f}(x)\right) = \underline{f}(x) \leq f(x),\; x \in X,\; n \in \mathbb{N}.\end{aligned}$$

It remains to prove that $f = \sup_{n\in\mathbb{N}} \xi_n$. We will show first that $\underline{f}$ is the point-wise supremum of the sequence $(\xi_n)_{n\in\mathbb{N}}$, that is,

$$\underline{f}(x) = \sup_{n\in\mathbb{N}}(\xi_n(x)),\; x \in X. \tag{41}$$

Let $x \in X$ and let $\epsilon > 0$ be arbitrary. Using that the function $h \circ \underline{f}$ is lower semi-continuous there exists $\nu > 0$ such that $h(\underline{f}(t)) > h(\underline{f}(x)) - \epsilon$ whenever $\rho(t, x) < \nu$. Let $m \in \mathbb{N}$ be such that $m \geq \dfrac{h(\underline{f}(x)) - \epsilon + 1}{\nu}$. It is easy to see that

$$\varphi_n(t, x) \geq h(\underline{f}(x)) - \epsilon\;,\; t \in X,\; n \geq m. \tag{42}$$

Indeed,

$$\text{if } \rho(t,x) \geq \nu \text{ then } \varphi_n(t,x) > -1+n\nu \geq -1+\frac{h(\underline{f}(x))-\epsilon+1}{\nu}\nu = h(\underline{f}(x))-\epsilon;$$

$$\text{if } \rho(t,x) < \nu \text{ then } \varphi_n(t,x) \geq h(\underline{f}(x))-\epsilon+n\rho(t,x) \geq h(\underline{f}(x))-\epsilon.$$

Using (42) for $n \geq m$ we have

$$\psi_n(x) = \inf_{t\in X} \varphi_n(t,x) \geq h(\underline{f}(x))-\epsilon.$$

Therefore

$$\sup_{n\in\mathbb{N}}(\psi_n(x)) \geq h(\underline{f}(x))-\epsilon.$$

Since ϵ in the above inequality is arbitrary and using also (36) we obtain

$$\sup_{n\in\mathbb{N}}(\psi_n(x)) = h(\underline{f}(x)).$$

The function h^{-1} used in the definition of ξ_n, see (40), is continuous and strictly increasing. Then we have

$$\sup_{n\in\mathbb{N}}(\xi_n(x)) = \sup_{n\in\mathbb{N}}(h^{-1}(\psi_n(x))) = h^{-1}\sup_{n\in\mathbb{N}}(\psi_n(x)) = h^{-1}(h(\underline{f}(x))) = \underline{f}(x),$$

which proves (41). Finally using Theorem 24 and (21) it follows from (41) that

$$\sup_{n\in\mathbb{N}}\xi_n = F(S(\underline{f})) = F(\overline{f}) = f.$$

This completes the proof. □

Acknowledgements. The authors would like to thank Prof. J.J. Grobler and Prof. E.E. Rosinger for their useful suggestions and comments. Thanks are also addressed to the National Research Foundation of South Africa for a research grant within the focus area Unlocking the Future.

REFERENCES

1. G. ALEFELD AND J. HERZBERGER, *Introduction to Interval Computations,* Academic Press, New York, 1983.
2. R. ANGUELOV, Dedekind order completion of C(X) by Hausdorff continuous functions, *Quaestiones Mathematicae* **27** (2004), 153–170.
3. R. ANGUELOV, S. MARKOV AND B. SENDOV, *On the Normed Linear Space of Hausdorff Continuous Functions,* Lecture Notes in Computer Science, to appear.
4. R. ANGUELOV AND E.E. ROSINGER, Hausdorff continuous solutions of nonlinear partial differential equations through the order completion method, Quaestiones Mathematicae, **28**(3) (2005), 271–285.
5. R. BEATTIE AND H.-P. BUTZMANN, *Convergence structures and applications to functional analysis,* Kluwer Academic Plublishers, Dordrecht - Boston - London, 2002.

6. G. BIRKHOFF, *Lattice Theory,* AMS, Providence, Rhode Island, 1973.

7. S. GÄHLER, W. GÄHLER AND G. KNEIS, Completion of pseudo-topological vector spaces, *Mathematische Nachrichten* bf 75 (1976), 185–206.

8. A. KRIEGL AND P.W. MICHOR, The convenient Setting of Global Analysis, *Mathematical Serveys Monographs,* Vol. 53, AMS, Providence, Rhode Island, 1997.

9. W.A. LUXEMBURG AND A.C. ZAANEN, *Riesz Spaces I,* North-Holland, Amsterdam - London, 1971.

10. S. MARKOV, Calculus for interval functions of a real variable, *Computing* **22** (1979), 325–337.

11. R.A. MCCOY AND I. NTANTU, *Topological Properties of Spaces of Continuous Functions,* Springer-Verlag, Berlin - Heidelberg, 1988.

12. V. OLEJČEK, Order Convergence and Order Topology on a Poset, *Int. J. of Theoretical Physics* **38**(2) (1999), 557–561.

13. A. PERESSINI, *Ordered topological vector spaces,* Harper & Row, New York - Evanston - London, 1967.

14. B. SENDOV, *Hausdorff approximations,* Kluwer Academic, Boston, 1990.

15. J.H. VAN DER WALT, *Order convergence in sets of Hausdorff continuous functions,* Honors Essay, University of Pretoria, 2004.

16. A.C. ZAANEN, *Introduction to Operator Theory in Riesz Spaces,* Springer, Berlin - Heidelberg - New York, 1997.

Received 8 April, 2005.