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DEDEKIND ORDER COMPLETION OF $C(X)$ BY HAUSDORFF CONTINUOUS FUNCTIONS

R. Anguelov

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa.
E-Mail anguelov@scientia.up.ac.za

Abstract. The concept of Hausdorff continuous interval valued functions, developed within the theory of Hausdorff approximations and originaly defined for interval valued functions of one real variable is extended to interval valued functions defined on a topological space X. The main result is that the set $\mathbb{H}_{ft}(X)$ of all finite Hausdorff continuous functions on any topological space X is Dedekind order complete. Hence it contains the Dedekind order completion of the set $C(X)$ of all continuous real functions defined on X as well as the Dedekind order completion of the set $C_b(X)$ of all bounded continuous functions on X. Under some general assumptions about the topological space X the Dedekind order completions of both $C(X)$ and $C_b(X)$ are characterised as subsets of $\mathbb{H}_{ft}(X)$. This solves a long outstanding open problem about the Dedekind order completion of $C(X)$. In addition, it has major applications to the regularity of solutions of large classes of nonlinear PDEs.

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1. Introduction. The fact that the set $C(X)$ of all continuous real valued functions on a topological space X is generally not Dedekind order complete with respect to the point-wise defined partial order

$$f \le g \Longleftrightarrow f(x) \le g(x),\ x \in X. \tag{1}$$

is well known and can be shown by trivial examples. We consider the problem of constructing a Dedekind order completion of $C(X)$ through functions defined on the same space X. An earlier result by Dilworth, see [5], gives a Dedekind order completion of the set $C_b(X)$ of all bounded continuous functions on a completely regular topological space X through the so called normal upper semi-continuous functions on X, the Dedekind order completion of $C(X)$ in the general case when $C(X)$ contains unbounded functions remaining an open problem. Here we obtain Dedekind order completions of both $C_b(X)$ and $C(X)$ through Hausdorff continuous interval valued functions. The significance of this result is emphasized by the fact that important applications involve sets of continuous functions which are not bounded. In [10] it was shown that arbitrary nonlinear PDEs defined by continuous, not necessarily smooth or analytic expressions, have solutions that can be assimilated with Lebesgue measurable functions. This powerful earlier existence results can now significantly be improved with respect to the regularity of solutions by showing that the solutions are in fact Hausdorff continuous, see for details [3].

We recall that a partially ordered set P is called Dedekind order complete if every subset of P which is bounded from above has a supremum in P and every subset of P which is bounded from below has an infimum in P. A general result on the Dedekind order completion of partially ordered sets was established by MacNeilly in 1937 (see [7] for a more recent presentation). The problem of the order completion of $C(X)$ is particularly addressed in [8]. As stated above, the problem of constructing a Dedekind order completion of $C(X)$ as a set of functions on X was partially addressed by Dilworth. More precisely, it was proved in [5] that if X is completely regular, then the Dedekind order completion of $C_b(X)$ is isomorphic with the lattice of the normal upper semi-continuous functions defined on X. Our approach is to consider $C(X)$ as a subset of the set of interval valued functions defined on X and find the Dedekind order completion of $C(X)$ within this set.

Denote by $\mathbb{I}\,\overline{\mathbb{R}}$ the set of all usual or extended real intervals

$$\mathbb{I}\,\overline{\mathbb{R}} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\},\ \underline{a} \le \overline{a}\}.$$

We consider on $\mathbb{I}\,\overline{\mathbb{R}}$ the partial order $\le$ defined in [9] through

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Longleftrightarrow \underline{a} \le \underline{b},\ \overline{a} \le \overline{b}. \tag{2}$$

In the set of interval valued functions $\mathbb{A}(X) = \{f : X \to \mathbb{I}\,\overline{\mathbb{R}}\}$, a partial order is induced by (2) in a point-wise way similar with (1). Identifying $x \in \overline{\mathbb{R}}$ with $[x, x] \in \mathbb{I}\,\overline{\mathbb{R}}$ we consider $\overline{\mathbb{R}}$ as a subset of $\mathbb{I}\,\overline{\mathbb{R}}$. In this way $\mathbb{A}(X)$ contains the set of extended real valued functions $\mathcal{A}(X) = \{f : X \to \overline{\mathbb{R}}\}$. Hence

$$C(X) \subset \mathcal{A}(X) \subset \mathbb{A}(X).$$

In Section 2 we define the set $\mathbb{H}_{ft}(X)$ of all the Hausdorff continuous interval valued functions with values finite real intervals, in which case we shall have

$$C(X) \subset \mathbb{H}_{ft}(X) \subset \mathbb{A}(X). \tag{3}$$

In Sections 3-5 the following results are presented:

1. The set $\mathbb{H}_{ft}(X)$ is Dedekind order complete. Hence it contains the Dedekind order completions $C(X)^{\#}$ and $C_b(X)^{\#}$ of $C(X)$ and $C_b(X)$, respectively.
2. For a completely regular topological space X

   $$C_b(X)^{\#} = \mathbb{H}_b(X) \subseteq \mathbb{H}_{ft}(X)$$

   where $\mathbb{H}_b(X)$ is the set of all bounded Hausdorff continuous functions defined on X.

3. For a completely regular topological space X

$$C(X)^{\#} = \mathbb{H}_{cm}(X) \subseteq \mathbb{H}_{ft}(X)$$

where $\mathbb{H}_{cm}(X)$ is the set of all Hausdorff continuous functions with continuous majorant and continuous minorant on X.

4. If X is a metric space then

$$C(X)^{\#} = \mathbb{H}_{cm}(X) = \mathbb{H}_{ft}(X).$$

The characterization of the topological spaces X for which $\mathbb{H}_{cm}(X) = \mathbb{H}_{ft}(X)$ remains an open problem.

Historically, the interval analysis, or the analysis of interval valued functions, is associated with, so called, validated computing where algorithms generating validated bounds for the exact solutions of mathematical problems are designed and investigated [1], [6]. However, interest in the interval valued functions comes also from other branches of mathematics such as nonlinear partial differential equations, see [3] which strengthens the results in [10], and approximation theory [11]. In fact, Hausdorff continuous functions of one real variable were first introduced by Sendov [11] in connection with Hausdorff approximations of real functions of real argument. The name Hausdorff continuous is due to the characterization of these functions in terms of the Hausdorff distance between the graphs of real functions as defined in [11]. The concept was further developed in [2] as part of the analysis of interval valued functions. Here, as a new departure in applying the ideas of interval analysis, we consider the concept of Hausdorff continuity for interval valued functions defined on a topological space X and apply it to the Dedekind order completion of $C_b(X)$ and $C(X)$.

2. Hausdorff continuous interval valued functions.

For every $x \in X$, $\mathcal{V}_x$ denotes the set of neighborhoods of x. We consider, [4], the pair of mappings $I : \mathbb{A}(X) \to \mathcal{A}(X)$, $S : \mathbb{A}(X) \to \mathcal{A}(X)$, called lower Baire, and upper Baire operators, respectively, where for every function $f \in \mathbb{A}(X)$ and $x \in X$, we have

$$I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf\{z \in f(y) : y \in V\}, \tag{4}$$

$$S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup\{z \in f(y) : y \in V\}. \tag{5}$$

The operator $F : \mathbb{A}(X) \to \mathbb{A}(X)$ defined by

$$F(f)(x) = [I(f)(x), S(f)(x)],\ f \in \mathbb{A}(X),\ x \in X,$$

is called graph completion.

Let us note that the lower Baire operator $I : f \to I(f)$, the upper Baire operator $S : f \to S(f)$ and the graph completion operator $F : f \to F(f) = [I(f), S(f)]$ are all monotone with respect to the order $\leq$ in $\mathbb{A}(X)$, which means that for every two functions $f, g \in \mathbb{A}(X)$ we have

$$f \leq g \Longrightarrow I(f) \leq I(g),\ S(f) \leq S(g),\ F(f) \leq F(g). \tag{6}$$

The operator F is also monotone about inclusion

$$f(x) \subseteq g(x), x \in X \Longrightarrow F(f)(x) \subseteq F(g)(x),\ x \in X.$$

Furthermore, all three operators are idempotent, i.e. for every $f \in \mathbb{A}(X)$

$$I(I(f)) = I(f),\ S(S(f)) = S(f),\ F(F(f)) = F(f). \tag{7}$$

The fixed points of the operators I and S are the lower and upper semi-continuous functions, respectively, defined on X. Let us recall the definitions of lower and upper semi-continuity.[4]

Definition 1. A function $f : X \to \overline{\mathbb{R}}$ is called lower semi-continuous at $x \in X$ if for every $m < f(x)$ there exists $V \in \mathcal{V}_x$ such that $m < f(y)$ for all $y \in V$. If $f(x) = -\infty$, then f is assumed lower semi-continuous at x.

Definition 2. A function $f : X \to \overline{\mathbb{R}}$ is called upper semi-continuous at $x \in X$ if for every $m > f(x)$ there exists $V \in \mathcal{V}_x$ such that $m > f(y)$ for all $y \in V$. If $f(x) = +\infty$, then f is assumed upper semi-continuous at x.

Definition 3. A function $f : X \to \overline{\mathbb{R}}$ is called lower (upper) semi-continuous on X if it is lower (upper) semi-continuous at every point of X.

It is easy to see that for every $f \in \mathbb{A}(X)$ the functions $I(f)$ and $S(f)$ are, respectively, lower and upper semi-continuous on X. Furthermore, if $f \in \mathcal{A}(X)$ then

$$f \text{ - lower semi-continuous on } X \Longleftrightarrow I(f) = f,$$
$$f \text{ - upper semi-continuous on } X \Longleftrightarrow S(f) = f.$$

Definition 4. A function $f \in \mathbb{A}(X)$ is called Hausdorff continuous, or H-continuous, if for every function $g \in \mathbb{A}(X)$ which satisfies the inclusion $g(x) \subseteq f(x)$, $x \in X$, we have $F(g)(x) = f(x)$, $x \in X$.

We denote by $\mathbb{H}(X)$ the subset of $\mathbb{A}(X)$ consisting of all the H-continuous functions while by $\mathbb{H}_{ft}(X)$ we denote the set of all H-continuous functions on X which assume finite values for every $x \in X$, that is,

$$\mathbb{H}_{ft}(X) = \{f \in \mathbb{H}(X) : f(x) \subseteq \mathbb{R},\ x \in X\}.$$

It is easy to see that all usual continuous real-valued functions defined on X are H-continuous. Indeed, since a continuous function f is both lower and upper semi-continuous we have

$$F(f) = [I(f), S(f)] = [f, f] = f.$$

Furthermore, if $g \in \mathbb{A}(X)$ is such that $g(x) \subseteq f(x)$, $x \in X$, then $g(x) = f(x)$, $x \in X$, because $f(x)$ is a point interval for all $x \in X$. Hence $F(g)(x) = F(f)(x) = f(x)$, $x \in X$, and the H-continuity of f follows from the Definition 4 above. Thus

$C(X) \subseteq \mathbb{H}(X)$. Moreover, since the functions in $C(X)$ assume values which are finite real numbers we have

$$C(X) \subseteq \mathbb{H}_{ft}(X) \subseteq \mathbb{H}(X).$$

We will show in the next section that the set $\mathbb{H}_{ft}(X)$ is Dedekind order complete. Therefore, in view of the above inclusion, for the characterization of the Dedekind order completion of $C(X)$ we only need to consider $\mathbb{H}_{ft}(X)$.

Let us note that the set $\mathbb{H}_{ft}(X)$ is certainly wider than $C(X)$. Here are some examples of H-continuous functions which are *not* continuous.

EXAMPLE 1. Let $X = \mathbb{R}$. For $x \in X$

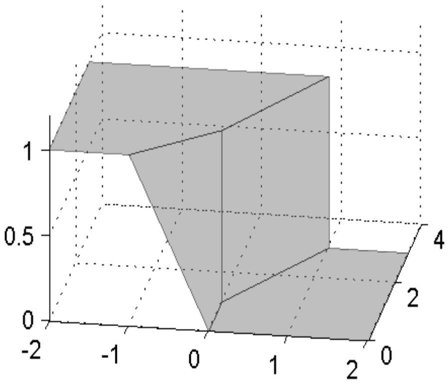
$$s(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ [-1,1] & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}.$$

EXAMPLE 2. Let $X = \mathbb{R}^2$. For $x \in X$

$$f(x) = \begin{cases} s\left(\sin \frac{1}{||x||}\right) & \text{if} \quad x \neq (0,0) \\ [-1,1] & \text{if} \quad x = (0,0) \end{cases}.$$

EXAMPLE 3. Consider a typical example of solution of a nonlinear shock wave equation. Here $X = \{(x,t) : t \geq 0\} \subseteq \mathbb{R}^2$ and for $(x,t) \in X$

$$y(x,t) = \begin{cases} 1 & \text{if} \quad t \in [0,1), x < t-1 \\ \frac{x}{t-1} & \text{if} \quad t \in [0,1), x \in [t-1,0] \\ 0 & \text{if} \quad t \in [0,1), x > 0 \\ 1 & \text{if} \quad t \geq 1\,, \quad x < \frac{t-1}{2} \\ [-1,1] & \text{if} \quad t \geq 1\,, \quad x = \frac{t-1}{2} \\ 0 & \text{if} \quad t \geq 1\,, \quad x > \frac{t-1}{2} \end{cases}$$



Let $f \in \mathbb{A}(X)$. For every $x \in X$ the value of f is an interval $[\underline{f}(x), \overline{f}(x)]$. Hence, the function f can be written in the form $f = [\underline{f}, \overline{f}]$ where $\underline{f}, \overline{f} \in \mathcal{A}(X)$ and $\underline{f} \leq \overline{f}$. The lower and upper Baire operators of the interval valued function f can be conveniently represented in terms of the functions $\underline{f}$ and $\overline{f}$. Indeed, from (4) and (5) it is easy to see that

$$I(f) = I(\underline{f}) \quad \text{and} \quad S(f) = S(\overline{f}). \tag{8}$$

Hence $F(f)$ can be written in the form

$$F(f) = [I(\underline{f}), S(\overline{f})]. \tag{9}$$

Therefore,

$$F(f) = f \Longleftrightarrow \underline{f} = I(\underline{f}),\ \overline{f} = S(\overline{f}) \Longleftrightarrow \left\{ \begin{array}{l} \overline{f} \text{ - upper semi-continuous} \\ \underline{f} \text{ - lower semi-continuous} \end{array} \right. . \tag{10}$$

It is easy to see that if f is H-continuous then it is a fixed point of the operator F. Indeed, from the Definition 4 and the inclusion $f(x) \subseteq f(x)$, $x \in X$, it follows that $F(f) = f$. Thus, in view of (10), we have that if $f = [\underline{f}, \overline{f}]$ is H-continuous then the functions $\underline{f}$, $\overline{f}$ are lower and upper semi-continuous functions, respectively. The following theorem provides convenient criteria for recognizing H-continuous functions. These criteria are discussed in [11]. However, since there only the case $X \subseteq \mathbb{R}$ is considered, we provide here a short proof.

Theorem 1. *Let $f = [\underline{f}, \overline{f}] \in \mathbb{A}(X)$. The following conditions are equivalent:*
a) The function f is H-continuous.
b) $F(\underline{f}) = F(\overline{f}) = f$.
c) $S(\underline{f}) = \overline{f}$, $I(\overline{f}) = \underline{f}$.

Proof. a)$\Longrightarrow$b): Since f is H-continuous using the Definition 4 we obtain

$$\underline{f}(x) \subseteq f(x), x \in X \Longrightarrow F(\underline{f})(x) = F(f)(x) = f(x),\ x \in X.$$

In a similar way we prove $F(\overline{f}) = f$.

b)$\Longrightarrow$c): We have

$$[I(\underline{f}), S(\underline{f})] = F(\underline{f})) = f = [\underline{f}, \overline{f}] \Longrightarrow S(\underline{f}) = \overline{f},$$
$$[I(\overline{f}), S(\overline{f})] = F(\overline{f})) = f = [\underline{f}, \overline{f}] \Longrightarrow I(\overline{f}) = \underline{f}.$$

c)$\Longrightarrow$a): Let us assume that $g = [\underline{g}, \overline{g}] \in \mathbb{A}(X)$ is such that $g(x) \subseteq f(x),\ x \in X$. Using the monotonicity of the operators I and S, thus from the inequalities

$$\underline{f} \le \underline{g} \le \overline{g} \le \overline{f}$$

it follows that

$$S(\underline{f}) \le S(\overline{g}) \le S(\overline{f}) = \overline{f},$$
$$\underline{f} = I(\underline{f}) \le I(\underline{g}) \le I(\overline{f}).$$

The equalities in c) imply that $S(\overline{g}) = \overline{f}$ and $I(\underline{g}) = \underline{f}$. Therefore, $F(g) = f$, which shows that f is H-continuous. □

Theorem 2. *Let $f \in \mathbb{A}(X)$. Both functions $F(S(I(f)))$ and $F(I(S(f)))$ are H-continuous.*

Proof. Denote $g = [\underline{g}, \overline{g}] = F(S(I(f))$. Clearly the functions $\underline{g} = I(S(I(f)))$ and $\overline{g} = S(I(f))$ are lower and upper semi-continuous, respectively. Using the monotonicity of the operators I and S we obtain the inequalities

$$S(\underline{g}) = S(I(S(I(f)))) \geq S(I(I(f))) = S(I(f)) = \overline{g},$$
$$S(\underline{g}) \leq S(\overline{g}) = \overline{g},$$

which imply $S(\underline{g}) = \overline{g}$. Furthermore, we have

$$I(\overline{g}) = I(S(I(f))) = \underline{g}.$$

Then, it follows from Theorem 1 (a and c) that function $g = F(S(I(f))$ is H-continuous. The H-continuity of $F(I(S(f)))$ is proved in a similar way. □

THEOREM 3. *Let $f = [\underline{f}, \overline{f}]$ be an H-continuous function on X.*
a) If $\underline{f}$ or $\overline{f}$ is continuous at a point $a \in X$ then $\underline{f}(a) = \overline{f}(a)$.
b) If $\underline{f}(a) = \overline{f}(a)$ for some $a \in X$ then both $\underline{f}$ and $\overline{f}$ are continuous at a.

Proof. a) Let $\underline{f}$ be continuous at $a \in X$. Assume that the equality at a) does not hold. Thus, $\underline{f}(a) < \overline{f}(a)$. Then, since $\underline{f}$ is lower semi-continuous at a there exists $V_1 \in \mathcal{V}_a$ such that $\underline{f}(y) < \frac{1}{2}(\underline{f}(a) + \overline{f}(a))$ whenever $y \in V_1$. Hence

$$\begin{aligned} S(\underline{f})(a) &= \inf_{V \in \mathcal{V}_a} \sup\{\underline{f}(y) : y \in V\} \leq \sup\{\underline{f}(y) : y \in V_1\} \\ &\leq \frac{1}{2}(\underline{f}(a) + \overline{f}(a)) < \overline{f}(a). \end{aligned}$$

This, according to Theorem 1 (a and c), implies that f is not H-continuous which contradicts the condition of the theorem. Thus, $\underline{f}(a) = \overline{f}(a)$. The case when $\overline{f}$ is continuous is treated in the same way.

b) Let $\varepsilon > 0$. Using the semi-continuity of $\underline{f}$ and $\overline{f}$ we obtain that there exists a $V_2 \in \mathcal{V}_a$ such that for every $y \in V_2$ we have

$$\overline{f}(y) \geq \underline{f}(y) > \underline{f}(a) - \varepsilon = f(a) - \varepsilon,$$
$$\underline{f}(y) \leq \overline{f}(y) < \overline{f}(a) + \varepsilon = f(a) + \varepsilon.$$

Therefore, $|\underline{f}(y) - \underline{f}(a)| = |\underline{f}(y) - f(a)| < \varepsilon$ and $|\overline{f}(y) - \overline{f}(a)| = |\overline{f}(y) - f(a)| < \varepsilon$ for all $y \in V_2$, which implies that both $\underline{f}$ and $\overline{f}$ are continuous at a. □

The following theorem shows a similarity between continuous functions and H-continuous functions.

THEOREM 4. *Let f, g be H-continuous on X and let D be a dense subset of X. Then*

$$a) \quad f(x) \leq g(x),\ x \in D \Longrightarrow f(x) \leq g(x),\ x \in X,$$
$$b) \quad f(x) = g(x),\ x \in D \Longrightarrow f(x) = g(x),\ x \in X.$$

Proof. a) Let $f = [\underline{f}, \overline{f}]$ and let $g = [\underline{g}, \overline{g}]$. Let also $x \in X$ and V_1, V_2 be arbitrary neighborhoods of x. Since D is dense in X there exists $z_0 \in D \cap (V_1 \cap V_2)$, that is $z_0 \in D \cap V_1$ and $z_0 \in D \cap V_2$. Therefore

$$\inf\{\underline{f}(y) : y \in V_1\} \le \underline{f}(z_0) \le \underline{g}(z_0) \le \overline{g}(z_0) \le \sup\{\overline{g}(y) : y \in V_2\}$$

Using that V_1 and V_2 are chosen independently we have

$$I(\underline{f})(x) = \sup_{V_1 \in \mathcal{V}_x} \inf\{\underline{f}(y) : y \in V_1\} \le \inf_{V_2 \in \mathcal{V}_x} \sup\{\overline{g}(y) : y \in V_2\} = S(\overline{g})(x).$$

Since $\underline{f}$ and $\overline{g}$ are lower and upper semi-continuous functions, respectively, the above inequality implies

$$\underline{f}(x) \le \overline{g}(x),\ x \in X. \tag{11}$$

Furthermore, $S(\underline{f}) = \overline{f}$ and $I(\overline{g}) = \underline{g}$, because both f and g are H-continuous, see Theorem 1. Hence, using the monotonicity of the operators I and S, see (6), and inequality (11) we obtain

$$\overline{f} = S(\underline{f}) \le S(\overline{g}) = \overline{g},$$
$$\underline{f} = I(\underline{f}) \le I(\overline{g}) = \underline{g}.$$

Therefore, $f \le g$ on X.

b) The proof follows immediately from a) because $f(x) = g(x)$ means that both relations $f(x) \le g(x)$ and $f(x) \ge g(x)$ are satisfied. □

3. Dedekind order completeness of $\mathbb{H}_{ft}(X)$. In this section we will prove that $\mathbb{H}_{ft}(X)$ is Dedekind order complete, see Theorems 5 and 6. Here we recall that the partial order on $\mathbb{H}_{ft}(X)$ is the one induced by the partial order on $\mathbb{A}(X)$. And the partial order on $\mathbb{A}(X)$ is induced by (2) point-wise on the respective functions similar with (1).

Upon an obvious extension of the respective result in [4] we have the following lemma about semi-continuous functions.

LEMMA 1. *a) Let $L \subseteq \mathcal{A}(X)$ be a set of lower semi-continuous functions. Then the function l defined by $l(x) = \sup\{f(x) : f \in L\}$ is lower semi-continuous.*
b) Let $U \subseteq \mathcal{A}(X)$ be a set of upper semi-continuous functions. Then the function u defined by $u(x) = \inf\{f(x) : f \in U\}$ is upper semi-continuous.

THEOREM 5. *Let $\mathcal{F}$ be a subset of $\mathbb{H}_{ft}(X)$ which is bounded from above. Then there exists $u \in \mathbb{H}_{ft}(X)$ such that $u = \sup \mathcal{F}$.*

Proof. Let $\psi \in \mathbb{H}_{ft}(X)$ be an upper bound of $\mathcal{F}$, that is, $f \le \psi$ for all $f \in \mathcal{F}$. Denote

$$g(x) = \sup\{\underline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{F}\},\ x \in X.$$

It is easy to see that

$$-\infty < g(x) \le \psi(x),\ x \in X. \tag{12}$$

The function g being a supremum of lower semi-continuous functions is also a lower semi-continuous function, see Lemma 1. Therefore, according to Theorem 2 we have $u = F(S(g)) = F(S(I(g))) \in \mathbb{H}(X)$. Furthermore, from the monotonicity of the operators S and F and the inequality (12) we have

$$u(x) = F(S(g))(x) \le F(S(\psi))(x) = \psi(x) < \infty,$$
$$u(x) = F(S(g))(x) \ge I(S(g))(x) \ge I(g)(x) = g(x) > -\infty.$$

Therefore, $u \in \mathbb{H}_{ft}(X)$. We will prove that u is the supremum of $\mathcal{F}$. More precisely, we will show that

A) u is an upper bound of $\mathcal{F}$, i.e. $f(x) \le u(x),\ \ x \in X,\ \ f \in \mathcal{F}$

B) u is the smallest upper bound of $\mathcal{F}$, i.e. for any function $h \in \mathbb{H}_{ft}(X)$,

$$f(x) \le h(x),\ \ x \in X,\ \ f \in \mathcal{F} \implies u(x) \le h(x),\ \ x \in X.$$

Using the monotonicity of the operators S and F, for every $f = [\underline{f}, \overline{f}] \in \mathcal{F}$ we have

$$f(x) \le \overline{f}(x) = S(\underline{f})(x) \le S(g)(x),\ x \in X,$$
$$f(x) = F(f)(x) \le F(S(g))(x) = u(x),\ x \in X,$$

i.e., u is an upper bound of the set $\mathcal{F}$.

Assume that function $h \in \mathbb{H}_{ft}(X)$ is such that

$$f(x) \le h(x),\ \ x \in X,\ \ f \in \mathcal{F}.$$

We have

$$g(x) = \sup\{I(f)(x) : f \in \mathcal{F}\} \le h(x),\ \ x \in X,$$

which implies $S(g) \le S(h)$. Hence $u = F(S(g)) \le F(S(h)) = h$. Therefore, $u = \sup \mathcal{F}$. □

In a similar manner we have

THEOREM 6. *Let $\mathcal{F}$ be a subset of $\mathbb{H}_{ft}(X)$ which is bounded from below. Then there exists $v \in \mathbb{H}(X)$ such that $v = \inf \mathcal{F}$.* □

4. Dedekind order completion of $C_b(X)$. From the Dedekind order completeness of $\mathbb{H}_{ft}(X)$ it follows that it contains the Dedekind order completion of its every subset, in particular the set $C_b(X)$. We will show that if X is completely regular the Dedekind order completion of $C_b(X)$ is

$$\mathbb{H}_b(X) = \{f \in \mathbb{H}(X) : \exists M \in \mathbb{R} : -M \le f(x) \le M,\ \ x \in X\}.$$

Since the inclusion

$$C_b(X) \subseteq \mathbb{H}_b(X) \subseteq \mathbb{H}_{ft}(X)$$

is obvious the statement will be proved through the next two theorems. More precisely, the Theorem 7 shows that the set $\mathbb{H}_b(X)$ is Dedekind order complete while the Theorem 8 shows that $\mathbb{H}_b(X)$ is the minimal Dedekind order complete set containing $C_b(X)$.

Theorem 7. *Let $\mathcal{F} \subseteq \mathbb{H}_b(X)$.*
a) If $\mathcal{F}$ is bounded from above then there exists $u \in \mathbb{H}_b(X)$ such that $u = \sup \mathcal{F}$.
b) If $\mathcal{F}$ is bounded from below then there exists $v \in \mathbb{H}_b(X)$ such that $v = \inf \mathcal{F}$.

Proof. a) Let $\mathcal{F}$ be bounded from above, that is, there exists a function $\psi \in \mathbb{H}_b(X)$ such that $f \le \psi$ for all $f \in \mathcal{F}$. Since $\mathbb{H}_{ft}(X)$ is Dedekind order complete the set $\mathcal{F}$ has a supremum in $\mathbb{H}_{ft}(X)$. Let $u = \sup \mathcal{F} \in \mathbb{H}_{ft}(X)$. We need to show that $u \in \mathbb{H}_b(X)$. Since ψ is an upper bound of $\mathcal{F}$, we have $u = \sup \mathcal{F} \le \psi$. Let $f_0 \in \mathcal{F}$. Then

$$f_0 \le u \le \psi.$$

Using that both f_0 and ψ are in $\mathbb{H}_b(X)$ there exist real numbers M_1 and M_2 such that

$$-M_1 \le f_0(x) \le M_1\ ,\ \ -M_2 \le \psi(x) \le M_2\ ,\ \ x \in X.$$

Hence, for $M = \max\{M_1, M_2\}$ we have

$$-M \le -M_1 \le f_0(x) \le u(x) \le \psi(x) \le M_2 \le M\ ,\ \ x \in X.$$

Therefore, $u \in \mathbb{H}_b(X)$.

The statement in b) is proved in a similar way. □

Theorem 8. *If the topological space X is completely regular, for every function $f \in \mathbb{H}_b(X)$ there exists a set $\mathcal{F} \subseteq C(X)$ such that $f = \sup \mathcal{F}$.*

Proof. Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}_b(X)$ be bounded by M, that is,

$$-M \le \underline{f}(x) \le \overline{f}(x) \le M,\ \ x \in X,$$

and let

$$\mathcal{F} = \{g \in C(X) : g \le f\}. \tag{13}$$

We will show that $f = \sup \mathcal{F}$. Let ε be an arbitrary positive number and let $x \in X$. Since the function $\underline{f}$ is lower semi-continuous, there exists a neighborhood $V_x \in \mathcal{V}_x$ such that

$$\underline{f}(y) > \underline{f}(x) - \varepsilon,\ \ y \in V_x. \tag{14}$$

Denote $m_x = \inf_{y \in V_x} \underline{f}(y)$. We have

$$\underline{f}(x) - \varepsilon \le m_x \le \underline{f}(y),\ \ y \in V_x. \tag{15}$$

Due to the complete regularity of X there exists a function ϕ_x such that

$$\begin{aligned} &\phi_x(x) = 1, \\ &\phi_x(y) = 0 \ \text{ if } \ y \notin V_x, \\ &\quad 0 \le \phi_x(y) \le 1 \text{ for all } y \in X. \end{aligned} \tag{16}$$

Consider the function g_x defined by

$$g_x(y) = (m_x + M)\phi_x(y) - M,\ y \in X.$$

Since ϕ_x is continuous then g_x is also continuous. Furthermore, using inequalities (15) and the properties (16) we have

$$\begin{aligned} g_x(y) &= (m_x + M) \times 0 - M = -M \le f(y) \ \text{ if } \ y \in X \backslash V_x, \\ g_x(y) &\le (m_x + M) \times 1 - M = m_x \le f(y) \ \text{ if } \ y \in V_x. \end{aligned}$$

Therefore, for every $x \in X$ the function g_x belongs to the set $\mathcal{F}$. Hence

$$(\sup \mathcal{F})(x) \ge g_x(x) = (m_x + M)\phi_x(x) - M = m_x \ge \underline{f}(x) - \varepsilon,\ x \in X,$$

where the last inequality follows from (15). Since the positive ε in the above inequality is arbitrary we have

$$(\sup \mathcal{F})(x) \ge \underline{f}(x),\ x \in X.$$

Using the monotonicity of the operator F and the fact that both $\sup \mathcal{F}$ and f are H-continuous we obtain

$$\sup \mathcal{F} \ge F(\underline{f}) = f. \tag{17}$$

However, f is an upper bound of $\mathcal{F}$, see (13), while $\sup \mathcal{F}$ is the smallest upper bound of $\mathcal{F}$ in $\mathbb{H}_b(X)$. Thus, from the inequality (17) it follows that $\sup \mathcal{F} = f$ which completes the proof. □

The Dedekind order completion $C_b(X)^\# = \mathbb{H}_b(X)$ of $C_b(X)$ discussed in this section is similar to the result of Dilworth in [5] where it is proved that $C_b(X)^\#$ is order isomorphic to the set of the so called normal upper semi-continuous functions. The approach presented here produces an alternative characterization of $C_b(X)^\#$ as a set of Hausdorff continuous functions. The power if this approach is further demonstrated in the next section by constructing the Dedekind order completion of the set $C(X)$ without any boundedness assumption about the functions, a case which is particularly important for applications to generalized solutions of nonlinear partial differential equations.

5. Dedekind order completion of $C(X)$. Since $C(X)$ is a subset of $\mathbb{H}_{ft}(X)$, see (3), which is Dedekind order complete, see Theorems 5 and 6, the Dedekind order completion of $C(X)$ is a subset of $\mathbb{H}_{ft}(X)$ for any topological space X. We will show that for a completely regular topological space X the Dedekind order completion of $C(X)$ is the set $\mathbb{H}_{cm}(X)$ of all Hausdorff continuous functions with a continuous majorant and a continuous minorant. More precisely,

$$\mathbb{H}_{cm}(X) = \{f \in \mathbb{H}_{ft}(X) : \exists \varphi, \psi \in C(X) : \varphi \le f \le \psi\}$$

and we will prove that

$$C(X)^\# = \mathbb{H}_{cm}(X). \tag{18}$$

We have the inclusion $C(X) \subseteq \mathbb{H}_{cm}(X)$. Similarly to the preceding section we will prove (18) by showing that $\mathbb{H}_{cm}(X)$ is Dedekind order complete, see Theorem 9, and that $\mathbb{H}_{cm}(X)$ is the minimal Dedekind order complete set which contains $C(X)$, see Theorem 10.

Theorem 9. *Let $\mathcal{F} \subseteq \mathbb{H}_{cm}(X)$.*
a) If $\mathcal{F}$ is bounded from above then there exists $u \in \mathbb{H}_{cm}(X)$ such that $u = \sup \mathcal{F}$.
b) If $\mathcal{F}$ is bounded from below then there exists $v \in \mathbb{H}_{cm}(X)$ such that $v = \inf \mathcal{F}$.

Proof. a) Let $\mathcal{F}$ be bounded from above, that is, there exists a function $h \in \mathbb{H}_{cm}(X)$ such that $f \le h$ for all $f \in \mathcal{F}$. Since $\mathbb{H}_{ft}(X)$ is Dedekind order complete the set $\mathcal{F}$ has a supremum in $\mathbb{H}_{ft}(X)$. Denote $u = \sup \mathcal{F} \in \mathbb{H}_{ft}(X)$. We need to show that $u \in \mathbb{H}_{cm}(X)$. Since h is an upper bound of $\mathcal{F}$, we have $u = \sup \mathcal{F} \le h$. Let $f_0 \in \mathcal{F}$. Then

$$f_0 \le u \le h.$$

Using that both f_0 and h are in $\mathbb{H}_{cm}(X)$ there exist real functions $\varphi_1, \varphi_2, \psi_1, \psi_2 \in C(X)$ such that

$$\varphi_1 \le f_0 \le \psi_1 \ , \ \ \varphi_2 \le h \le \psi_2.$$

Then, we have

$$\varphi_1 \le f_0 \le u \le h \le \psi_2.$$

Hence, $u \in \mathbb{H}_{cm}(X)$.

The statement in b) is proved in a similar way. □

Theorem 10. *If X is a completely regular topological space for every function $f \in \mathbb{H}_{cm}(X)$ there exists a set $\mathcal{F} \subseteq C(X)$ such that $f = \sup \mathcal{F}$.*

Proof. Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}_{cm}(X)$ and let $\varphi \in C(X)$ be a minorant of f, that is $\varphi \le f$. Consider the set

$$\mathcal{F} = \{g \in C(X) : g \le f\}. \tag{19}$$

Clearly, $\varphi \in \mathcal{F}$. Hence the set $\mathcal{F}$ is not empty. We will show that $f = \sup \mathcal{F}$. Let ε be an arbitrary positive number and let $x \in X$. Since the function $\underline{f}$ is lower semi-continuous, there exists a neighborhood $V_x \in \mathcal{V}_x$ such that

$$\underline{f}(y) > \underline{f}(x) - \varepsilon, \ y \in V_x. \tag{20}$$

Denote $m_x = \inf_{y \in V_x} \underline{f}(y)$. From (20) it follows that

$$\underline{f}(x) - \varepsilon \le m_x \le \underline{f}(y), \ y \in V_x. \tag{21}$$

Since the space X is completely regular there exists a function $\phi_x \in C(X)$ such that

$$\begin{aligned} &\phi_x(x) = 1, \\ &\phi_x(y) = 0 \ \text{ if } \ y \notin V_x, \\ &\quad 0 \le \phi_x(y) \le 1 \text{ for all } y \in X. \end{aligned} \tag{22}$$

Consider the function g_x defined by

$$g_x(y) = (m_x - \varphi(y))\phi_x(y) + \varphi(y),\ y \in X.$$

Since $\phi_x, \varphi \in C(X)$ then $g_x \in C(X)$. Furthermore, using inequalities (21) and the properties (22) we have

$$\begin{aligned}
&g_x(y) = (m_x - \varphi(y)) \times 0 + \varphi(y) = \varphi(y) \le f(y) \ \text{ if } \ y \in X \backslash V_x\ , \\
&g_x(y) \le 0 \times \phi_x(y) + \varphi(y) = \varphi(y) \le f(y) \ \text{ if } \varphi(y) > m_x\ , \\
&g_x(y) \le (m_x - \varphi(y)) \times 1 + \varphi(y) = m_x \le f(y) \ \text{ if } \ y \in V_x \text{ and } \varphi(y) < m_x\ .
\end{aligned}$$

Therefore, for every $x \in X$ we have $g_x \le f$ which implies $g_x \in \mathcal{F}$. Hence

$$(\sup \mathcal{F})(x) \ge g_x(x) = (m_x - \varphi(x))\varphi_x(x) + \varphi(x) = m_x \ge \underline{f}(x) - \varepsilon,\ x \in X,$$

where the last inequality follows from (21). Since the positive ε in the above inequality is arbitrary we have

$$(\sup \mathcal{F})(x) \ge \underline{f}(x),\ x \in X.$$

Using the monotonicity of the operator F and the fact that both $\sup \mathcal{F}$ and f are H-continuous we obtain

$$\sup \mathcal{F} \ge F(\underline{f}) = f. \tag{23}$$

However, f is an upper bound of $\mathcal{F}$, see (19), while $\sup \mathcal{F}$ is the smallest upper bound of $\mathcal{F}$ in $\mathbb{H}_{cm}(X)$. Thus, from the inequality (23) it follows that $\sup \mathcal{F} = f$ which completes the proof. □

The characterizations of the Dedekind order completion of $C(X)$ as the set $\mathbb{H}_{cm}(X)$ is interesting if the topological space X is such that $C(X)$ contains unbounded functions because the case of bounded continuous functions was dealt with in Section 4. For example, if X is a compact Hausdorff space, the elements of both $C(X)$ and $\mathbb{H}_{ft}(X)$ are bounded functions, that is, we have

$$C(X) = C_b(X) \text{ and } \mathbb{H}_b(X) = \mathbb{H}_{ft}(X),$$

which means that the Dedekind order completion of $C(X)$ is $\mathbb{H}_{ft}(X)$. Since $\mathbb{H}_{cm}(X) = C(X)^{\#}$, this also means that

$$\mathbb{H}_{cm}(X) = \mathbb{H}_{ft}(X). \tag{24}$$

The characterization of the topological spaces for which the condition (24) holds is still an open problem. However, we can show that (24) holds if X is an arbitrary metric space. Note that in this case $C(X)$ may contain unbounded functions.

Theorem 11. *Let (X, ρ) be a metric space. Then, every function $f \in \mathbb{H}_{ft}(X)$ has a continuous majorant and a continuous minorant, that is, $\mathbb{H}_{cm}(X) = \mathbb{H}_{ft}(X)$.*

Proof. We will use the function $h:\mathbb{R}\to(-1,1)\subset R$ defined by

$$h(z)=\frac{z}{1+|z|},\ z\in\mathbb{R}.$$

This real function is continuous and strictly increasing. The inverse function $h^{-1}:(-1,1)\to\mathbb{R}$ is given by

$$h^{-1}(z)=\frac{z}{1-|z|},\ z\in(-1,1),$$

and is also continuous and strictly increasing.

Let $f=[\underline{f},\overline{f}]\in\mathbb{H}_{ft}(X)$. Consider the function $\varphi:X\times X\to R$ defined by

$$\varphi(t,x)=h(\underline{f}(t))+\rho(t,x)=\frac{\underline{f}(t)}{1+|\underline{f}(t)|}+\rho(t,x). \tag{25}$$

It is easy to see that the function φ is bounded from below. Indeed, since the value of the metric ρ is always nonnegative and the fraction in (25) is greater than -1 we have $\varphi(t,x)>-1$. Then, we can define

$$\psi(x)=\inf\{\varphi(t,x):t\in X\}.$$

First we will show that the function ψ is continuous on X. From the triangular inequality of the metric ρ for every $x,y,t\in X$ we have

$$\rho(t,y)-\rho(x,y)\le\rho(t,x)\le\rho(t,y)+\rho(x,y).$$

Therefore

$$\varphi(t,y)-\rho(x,y)\le\varphi(t,x)\le\varphi(t,y)+\rho(x,y).$$

Taking the infimum on $t\in X$ we obtain

$$\psi(y)-\rho(x,y)\le\psi(x)\le\psi(y)+\rho(x,y).$$

Hence we have the inequality

$$|\psi(x)-\psi(y)|\le\rho(x,y),$$

which implies that the function ψ is continuous on X.

Our second step is to prove that ψ satisfies the inequalities

$$-1<\psi(x)\le\frac{\underline{f}(x)}{1+|\underline{f}(x)|}<1,\ x\in X. \tag{26}$$

For every $x\in X$ we have

$$\psi(x)=\inf\{\varphi(t,x):t\in X\}\le\varphi(x,x)=\frac{\underline{f}(x)}{1+|\underline{f}(x)|}.$$

Furthermore, since -1 is a lower bound of $\varphi(t,x)$ the inequality

$$\psi(x) \geq -1$$

also holds. It remains to prove that $\psi(x) \neq -1$. Let us assume that there exists $x \in X$ such that $\psi(x) = -1$. Let the real number μ be such that $-1 < \mu < \frac{\underline{f}(x)}{1+|\underline{f}(x)|}$. Then we have

$$h(\underline{f}(x)) = \frac{\underline{f}(x)}{1+|\underline{f}(x)|} > \mu > -1.$$

Due to the continuity of h there exists η such that

$$h(z) > \mu \text{ whenever } \underline{f}(x) - \eta < z < \underline{f}(x) + \eta.$$

Moreover, since h is strictly increasing we have

$$h(z) > \mu \text{ for all } z > \underline{f}(x) - \eta.$$

Using that the function $\underline{f}(t)$ is lower semi-continuous at x, there exists $\varepsilon > 0$ such that

$$\underline{f}(t) > \underline{f}(x) - \eta \quad \text{whenever} \quad \rho(t,x) < \varepsilon.$$

Hence

$$\frac{\underline{f}(t)}{1+|\underline{f}(t)|} = h(\underline{f}(t)) > \mu \quad \text{whenever} \quad \rho(t,x) < \varepsilon. \tag{27}$$

Let now $\delta = \min\{\varepsilon, \mu+1\}$. Since $\psi(x)$ is defined as an infimum on $t \in X$, there exists $t_\delta \in X$ such that

$$-1 = \psi(x) \leq \varphi(t_\delta, x) \leq \psi(x) + \delta = -1 + \delta$$

or, more precisely,

$$-1 \leq \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|} + \rho(t_\delta, x) \leq -1 + \delta.$$

Using simple manipulations we obtain

$$0 \leq \rho(t_\delta, x) \leq \delta - \left(1 + \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|}\right) < \delta \leq \varepsilon \tag{28}$$

$$-1 \leq \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|} \leq -1 + \delta \leq \mu. \tag{29}$$

The contradiction between inequalities (28), (29) on the one side and the condition (27) on the other side show that the assumption that $\psi(x) = -1$ for some $x \in X$ is false. Therefore $\psi(x) > -1$, $x \in X$.

Now we can consider the function

$$\phi(x) = h^{-1}(\psi(x)) = \frac{\psi(x)}{1-|\psi(x)|}, \ x \in X.$$

Due to inequalities (26) function ϕ is well defined for every $x \in X$. Furthermore, ϕ is continuous on X because ψ is continuous on X. Using the fact that the function h^{-1} is strictly increasing on the interval $(-1, 1)$ from the middle inequality in (26) we obtain

$$\phi(x) = h^{-1}(\psi(x)) \leq h^{-1}\left(\frac{\underline{f}(x)}{1+|\underline{f}(x)|}\right) = h^{-1}\left(h(\underline{f}(x)\right) = \underline{f}(x) \leq f(x).$$

Thus, ϕ is a continuous minorant of f. The existence of a continuous majorant is proved in a similar way. □

COROLLARY 1. *If X is a metric space then $C(X)^{\#} = \mathbb{H}_{ft}(X)$.*

6. Conclusion. This paper addresses the issue of finding a characterization for the Dedekind order completion of $C(X)$ in a constructive form, namely, as a set of functions on the same space X. Here X can be an arbitrary topological space. The functions, which give the completion, are the Hausdorff continuous functions, and as such, are in general interval valued functions. However, they are not unlike the usual real-valued functions, since interval values only appear at points of discontinuity. The set $\mathbb{H}_{ft}(X)$ of all finite Hausdorff continuous functions is Dedekind order complete and, thus, it contains the Dedekind completion of $C(X)$. It was shown that in the important case of X being a metric space $\mathbb{H}_{ft}(X)$ is the Dedekind order completion of $C(X)$. In the more general case of a completely regular topological space X the Dedekind order completion of $C(X)$ is the set $\mathbb{H}_{cm}(X)$ of all H-continuous functions with a continuous majorant and a continuous minorant. In the case considered by Dilworth [5], namely, the Dedekind order completion of $C_b(X)$,where X is a completely regular space, an alternative characterization of the Dedekind order completion of $C_b(X)$ was found in the form of the set $\mathbb{H}_b(X)$ - the set of all bounded Hausdorff continuous functions.

The Hausdorff continuous functions can be applied in further improving recent mathematical methods in the solution of large classes on nonlinear partial differential equations, methods which use the Dedekind order completion of $C(X)$, [10]. Let us note that the applicability of the set $\mathbb{H}_{ft}(X)$ in the order completion method discussed in [10] depends on the fact that, unlike the result in [5], there is no boundedness requirement on the functions. In addition to that, the Hausdorff continuous functions provide a convenient representation of nonlinear shock waves. For instance, the function in Example 3, as mentioned, is a solution of the inviscid Burger's, or nonlinear shock wave equation, with a shock along the line $t = 1 + 2x, x \geq 0$.

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